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ON THE REAL EXPONENTIAL FIELD WITH RESTRICTED ANALYTIC FUNCTIONS

BY

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ABSTRACT

The model-theoretic structure $(\mathbb{R}_{\text{an}}, \exp)$ is investigated as a special case of an expansion of the field of reals by certain families of C^∞-functions. In particular, we use methods of Wilkie to show that $(\mathbb{R}_{\text{an}}, \exp)$ is (finitely) model complete and O-minimal. We also prove analytic cell decomposition and the fact that every definable unary function is ultimately bounded by an iterated exponential function.

Introduction

Wilkie [W1,2] recently proved that the structure

$$\mathbb{R}_{\text{exp}} := (\mathbb{R}, <, 0, 1, +, -, \cdot, \exp)$$

is model complete, that is, $\text{Th}(\mathbb{R}_{\text{exp}})$ is model complete. Earlier it was shown, cf. [vdD2] and [D-vdD], that the structure

$$\mathbb{R}_{\text{an}} := (\mathbb{R}, <, 0, 1, +, -, \cdot, (\tilde{f})_{f \in \mathbb{R}\{X,m\}, m \in \mathbb{N}})$$

is model complete, where $\mathbb{R}\{X, m\} := \mathbb{R}\{X_1, \ldots, X_m\}$ denotes the ring of all power series in $X_1, \ldots, X_m$ over $\mathbb{R}$ that converge in a neighborhood of $[-1, 1]^m$, and where for $f \in \mathbb{R}\{X, m\}$ we define $\tilde{f}\colon \mathbb{R}^m \to \mathbb{R}$ by

$$\tilde{f}(x) := \begin{cases} f(x), & \text{for } x \in [-1,1]^m \\ 0, & \text{for } x \in \mathbb{R}^m \setminus [-1,1]^m. \end{cases}$$

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(The subscript "an" stands for "analytic"; by [D-vdD] the sets definable in $\mathbb{R}_{\text{an}}$ are exactly the *finitely subanalytic* sets introduced in [vdD2] .)

These two model completeness results imply, by older theorems of Łojasiewicz and Hovanskii, that $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{exp}}$ are O-minimal. This raises a question that we answer in this article as follows, cf. (6.9):

THEOREM: *$(\mathbb{R}_{\text{an}}, \exp)$ is model complete and O-minimal.*

(This makes $(\mathbb{R}_{\text{an}}, \exp)$ the largest presently known O-minimal expansion of the ordered field of reals.) The proof of this theorem is very much along the lines of Wilkie's proof for $\mathbb{R}_{\text{exp}}$: we extend arguments from [W1,2] to obtain a sort of *relativization* of Wilkie's theorem so that we can add the exponential function to suitable model complete and O-minimal expansions of the ordered field of reals, and *preserve* model completeness and O-minimality; see (6.10).

To check that this relativization applies to $\mathbb{R}_{\text{an}}$ we use a theorem of Frisch [F] to the effect that the power series ring $\mathbb{R}\{X, m\}$ is noetherian; see section 2. We also need an easy extension of Hovanskii's theorem [H]; see section 3. Another model complete expansion of the real field is

$$(\mathbb{R}, <, 0, 1, +, -, \cdot, \sin|[-\pi, \pi], \exp),$$

where, say, we define $\sin|[-\pi, \pi]$ to be 0 outside $[-\pi, \pi]$ to make it total. In fact, our relativization of Wilkie's theorem shows we can replace here $\sin|[-\pi, \pi]$ by the restrictions to $[0, 1]^m$ of any Pfaffian chain on $\mathbb{R}^m$, provided we put certain real constants in the language, cf. (6.12)(ii).

As mentioned already we follow here Wilkie's method for proving model completeness of certain expansions $\tilde{\mathbb{R}}$ of the field $\mathbb{R}$, which consists in carrying out the following three steps for $\tilde{T} := \text{Th}(\tilde{\mathbb{R}})$:

STEP 1: Show that $\tilde{T}$ is model complete if for each pair of models k and K of $\tilde{T}$ with $\mathsf{k} \subseteq \mathsf{K}$, every "regular" solution in K^n of a system of n equations in n unknowns given by terms over k lies in k^n. (See (1.1) below for the notion of **regular solution.**)

STEP 2: With k and K as in step 1, show that every regular solution in K^n of a system of n equations in n unknowns given by terms over k is k-bounded.

STEP 3: Improve the conclusion of step 2 by showing that each such regular solution actually lies in k^n.

This way of proving model completeness, in combination with the finiteness of the number of regular solutions (Hovanskii), has interesting consequences, the most important of which is O-minimality; cf. (5.11). Another is finite model completeness, cf. (5.14): Call an L-theory T **finitely model complete** if for each L-formula $\phi(x)$, $x = (x_1, \ldots, x_M)$, there is a quantifier free L-formula $\theta(x, y)$, with $y = (y_1, \ldots, y_N)$, such that $T \vdash \phi(x) \leftrightarrow \exists y\theta(x, y)$ together with an integer $k \geq 1$, such that $T \vdash \exists^{\leq k} y\theta(x, y)$. Finite model completeness implies of course model completeness, and we actually prove that all structures above are finitely model complete.

Finite model completeness is a weaker variant of the strong model completeness of [vdD3]. Related to finite model completeness and O-minimality are some further results for $\mathbb{R}_{\exp}$, $(\mathbb{R}_{\mathrm{an}}, \exp)$ and similar expansions $\tilde{\mathbb{R}}$ of the field of reals discussed in this paper.

In section 7 we characterize the definable closure of a subset S in any structure K elementarily equivalent to $\tilde{\mathbb{R}}$ as the set of coordinates of regular solutions of systems of equations given by terms with constants from S.

In section 8 we prove that definable sets can be decomposed into finitely many *analytic* cells and that definable functions are piecewise analytic.

In section 9 we provide an iterated exponential bound for the asymptotic growth of functions $f\colon \mathbb{R} \to \mathbb{R}$ that are definable in $(\mathbb{R}_{\mathrm{an}}, \exp)$.

To be able to treat all the above expansions of the field of reals in an efficient and uniform way, we introduce in section 3 the notion of "system of C^∞-rings". Such a system consists for each $n \in \mathbb{N}$ of a ring of C^∞-functions on $\mathbb{R}^n$ closed under taking partial derivatives. A large part of sections 3 and 5 is devoted to deriving basic facts on such systems when they satisfy extra properties (Hovanskii property, noetherianity), and are extended in certain ways.

During our work on this paper Ressayre ([Re]) found a novel approach to proving model completeness of structures like $\mathbb{R}_{\exp}$ and $(\mathbb{R}_{\mathrm{an}}, \exp)$, leading to other results that seem hard to obtain via Wilkie's method, for instance, simple explicit axiomatizations of the theories of these structures relative to the corresponding "restricted theories". Elaborating on Ressayre's ideas one can show that the structure $(\mathbb{R}_{\mathrm{an}}, \exp)$ admits quantifier elimination when the language is extended only by a function symbol log to denote the logarithm function (defined to be 0, say, for nonpositive arguments). (See [vdD-M-M].) This in turn implies strong model completeness, instead of the finite model completeness of this paper. However, this approach does not give O-minimality without a certain amount of further work roughly equivalent to what is done in the present paper in sections 3 and 5. Also for the results in sections 7, 8 and 9 extra work would be needed, though it is fair to say these results are almost visibly evident from the Ressayre point of view. Nevertheless, it seems reasonable not to abandon the remarkable methods introduced by Wilkie, and to put them into a somewhat more general setting, as we have tried to do in this article.

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1. Preliminaries

1.1 DIFFERENTIABILITY AND REGULARITY. We often deal with C^∞-functions on ordered fields different from $\mathbb{R}$, so it may not be out of place to fix some definitions.

Let K *be any ordered field,* U *an open subset of* K^n. For $x = (x_1, \dots, x_n) \in \mathsf{K}^n$, put $|x| = \sup(|x_1|, \dots, |x_n|)$.

Consider a map $f = (f_1, \dots, f_m)\colon U \to \mathsf{K}^m$.

We call f **differentiable at the point** $a \in U$ if there is a K-linear map $T\colon \mathsf{K}^n \to \mathsf{K}^m$ such that for each $\epsilon > 0$ in K we have $|f(a+x)-(f(a)+T(x))| < \epsilon|x|$ for all sufficiently small vectors x in K^n. (Such a map is necessarily unique.) Clearly f is differentiable at a iff each of the components f_i is differentiable at a; then the partial derivatives $(\partial f_i/\partial x_j)(a)$ exist, these being defined by the usual ϵ-δ definition, with ϵ and δ ranging over K. If f is differentiable at a, then f is continuous at a, and the matrix of the linear map T as above relative to the standard bases is the m-by-n matrix $((\partial f_i/\partial x_j)(a))$, which we call the Jacobian matrix of f at a, and denote by $\partial(f_1, \dots, f_m)/\partial(x_1, \dots, x_n)(a)$. We call f a C^0-map if f is continuous, and inductively we define f to be a C^{k+1}-map ($k \geq 0$) if f is differentiable at each point $a \in U$ and the map $a \mapsto \partial(f_1, \dots, f_m)/\partial(x_1, \dots, x_n)(a)\colon U \to \mathsf{K}^{mn}$ is a C^k-map. Note that if f is C^{k+1}, then f is C^k. Finally, we call f a C^∞-map if f is a C^k-map for all $k \in \mathbb{N}$; in that case all partials of all orders of the f_i's exist and are continuous on

U. (The converse is true for $\mathsf{K} = \mathbb{R}$.) We leave to the reader the statement and (standard) proofs of the usual formal rules such as the chain rule for compositions of C^1-maps.

Assume now that the map f above is a C^∞-map and let $a \in U$.

We say that f is **regular at** a if the rank of the Jacobian matrix of f at a equals $\min\{m, n\}$. Otherwise (when the rank is less than $\min\{m, n\}$) we call a a **critical point of** f. We call $b \in \mathsf{K}^m$ a **regular value of** f if f is regular at every point $x \in U$ with $f(x) = b$. (In particular, all points $b \in \mathsf{K}^m \setminus f(U)$ are regular values of f.) For $b \in \mathsf{K}^m$ we put

$$\mathrm{Reg}(f, b) := \{x \in f^{-1}(b) \colon f \text{ is regular at } x\}.$$

Given $b = (b_1, \ldots, b_m) \in \mathsf{K}^m$ we also talk about the system of equations

$$\begin{gathered} f_1(x) = b_1 \\ \vdots \\ f_m(x) = b_m, \end{gathered}$$

and we express $x \in \mathrm{Reg}(f, b)$ also by saying that x is a **regular solution** of the system. When $b = (0, \ldots, 0)$ we also write $V(f_1, \ldots, f_m)$ or $V(f)$ for the zero set $f^{-1}(0)$ of $f_1, \ldots, f_m$, and $V^{\mathrm{reg}}(f_1, \ldots, f_m)$ or $V^{\mathrm{reg}}(f)$ for $\mathrm{Reg}(f, 0)$, the set of **regular zeros** of $f_1, \ldots, f_m$. Often we have $m = n$ (same number of equations as unknowns) and then we denote the determinant of the Jacobian matrix of f at a point $x \in U$ by $J(f)(x)$ or $J(f_1, \ldots, f_n)(x)$ and call this the **Jacobian of f at** x; note that then f is regular at x if and only if $J(f)(x) \neq 0$. (We remark that Wilkie writes $V^{\mathrm{ns}}(f_1, \ldots, f_m)$ instead of $V^{\mathrm{reg}}(f_1, \ldots, f_m)$ and speaks of nonsingular solutions instead of regular solutions.)

1.2 Definability and o-minimality. In this paper "**definable in a structure**" means "**definable in the structure using constants from the underlying set of the structure**", *unless indicated otherwise.* If A and B are L-structures for the same language L, we write $\mathsf{A} \subseteq \mathsf{B}$ to indicate that A is a substructure of B. Terms in a given language are often used to denote the functions they define in structures for that language, provided it is clear from context which structure is intended, and which cartesian power of the underlying set of the structure is the intended domain of definition of the function.

A structure $\mathsf{A} = (A, <, \dots)$ with a distinguished linear order $<$ on its underlying set A is called **O-minimal** if $(A, <)$ is dense without endpoints and each subset of A that is definable in A is a finite union of intervals (a, b) and points, where $a, b \in A \cup \{-\infty, +\infty\}$, $a < b$. Then every structure elementarily equivalent to A is also O-minimal, cf. [K-P-S], and therefore we may (and shall) call the complete theory $\mathrm{Th}(\mathsf{A})$ O-minimal.

Let $\tilde{\mathbb{R}}$ be an O-minimal expansion of the structure $(\mathbb{R}, <, 0, 1, -, +, \cdot)$, and put $\tilde{T} := \mathrm{Th}(\tilde{\mathbb{R}})$, an extension of RCF, the theory of ordered real closed fields. Then $\tilde{T}$ has definable Skolem functions, for the same reason that RCF has, cf. [vdD1]. Let $\mathsf{K} \models \tilde{T}$.

By definability of Skolem functions, the definable closure of any subset of K is (the underlying set of) an elementary submodel of K. Moreover, the operation of taking the definable closure (in K) of subsets of K is a closure operation satisfying the Steinitz exchange property, cf. [P-S], and hence gives rise to a notion of **rank**. In particular, given elementary submodels k_1 and k_2 of K with $\mathsf{k}_1 \subseteq \mathsf{k}_2$ we write $\mathrm{rk}(\mathsf{k}_2|\mathsf{k}_1)$ for the cardinality of any non-redundant set of generators of k_2 over k_1, "set of generators" to be taken in the sense of the definable closure operation, and "non-redundant" meaning that each strictly smaller set has a strictly smaller definable closure. (In the case $\tilde{T} = \mathrm{RCF}$, this rank is just transcendence degree.)

Given a definable C^∞-map $f\colon \mathsf{K}^n \to \mathsf{K}^n$ and a point $b \in \mathsf{K}^n$ it follows from the implicit function theorem that all points of $\mathrm{Reg}(f, b) \subseteq \mathsf{K}^n$ are isolated in $\mathrm{Reg}(f, b)$, and since $\mathrm{Reg}(f, b)$ is a definable set, O-minimality implies:
$\mathrm{Reg}(f, b)$ *is finite, with a uniform finite bound on* $\mathrm{card}(\mathrm{Reg}(f, b))$ *as* b *ranges over* K^n.

1.3 Ordered Fields. Let K be an ordered field. We put $\mathrm{Pos}(\mathsf{K}) := \{x \in \mathsf{K}\colon x > 0\}$. Given a subfield k of K we say that $a \in \mathsf{K}$ is k-bounded if $|a| < b$ for some $b \in \mathrm{Pos}(\mathsf{k})$, and we say that a point $(a_1, \dots, a_n) \in \mathsf{K}^n$ is k-bounded if each coordinate a_i is k-bounded; the set of k-bounded elements of K is convex in K, and hence a valuation ring of K. For $\mathsf{k} = \mathbb{Q}$ = the prime field in K, we let $\mathrm{Fin}(\mathsf{K})$ denote the ring of $\mathbb{Q}$-bounded elements, as in [W2]. The valuation on K induced by the valuation ring $\mathrm{Fin}(\mathsf{K})$ is denoted by $\mathrm{ord}_{\mathsf{K}}$, or just ord if K is clear from context, and its value group by $\mathrm{ord}(\mathsf{K})$. (This is a slight abuse of notation, since $\mathrm{ord}(0) = \infty$ is not included in $\mathrm{ord}(\mathsf{K})$.) When k is a subfield of K we identify $\mathrm{ord}(\mathsf{k})$ with a subgroup of $\mathrm{ord}(\mathsf{K})$ in the usual way, and we consider $\mathrm{ord}_{\mathsf{K}}$ as an extension of $\mathrm{ord}_{\mathsf{k}}$. Note that if K is real closed, $\mathrm{ord}(\mathsf{K})$ is divisible, and hence a vector space over $\mathbb{Q}$ in a natural way. For real closed K we also use the "euclidean norm" $\|x\| := (x_1^2 + \cdots + x_n^2)^{1/2}$ for $x = (x_1, \ldots, x_n) \in \mathsf{K}^n$.

1.4 Smoothness and Rational Type. Let $\tilde{\mathbb{R}}$ be an expansion of the ordered field of reals and $\tilde{T} := \mathrm{Th}(\tilde{\mathbb{R}})$.

After Wilkie [W2] we call $\tilde{\mathbb{R}}$ (as well as $\tilde{T}$) **smooth** if the following three conditions are satisfied:

(S1) $\tilde{\mathbb{R}}$ is O-minimal;

(S2) for each model K of $\tilde{T}$ and each definable function $f\colon \mathsf{K} \to \mathsf{K}$ there is $n \in \mathbb{N}$ such that $|f(x)| \leq x^n$ for all sufficiently large x in K;

(S3) for each formula $\phi(x)$ in the language of $\tilde{T}$, $x = (x_1, \ldots, x_n)$, there are m, $p \in \mathbb{N}$ and C^∞-functions $F_i\colon \mathbb{R}^{n+m} \to \mathbb{R}$ for $i = 1, \ldots, p$, definable in $\tilde{\mathbb{R}}$ without constants from $\mathbb{R}$, such that

$$\tilde{\mathbb{R}} \models \forall x(\phi(x) \leftrightarrow \exists y(|y| \leq 1 \wedge \bigvee_i (N_i(y) \wedge F_i(x, y) = 0))),$$

where $y = (y_1, \ldots, y_m)$ and $N_i(y)$ is a conjunction of formulas $y_j \neq 0$.

(In (S2) it suffices to consider $\mathsf{K} = \tilde{\mathbb{R}}$, and functions definable in $\tilde{\mathbb{R}}$ without using real constants, but we shall not use this fact.)

We now have the following important result from Wilkie [W2]:

Proposition: *If $\tilde{T}$ is smooth and K is a model of $\tilde{T}$ of finite rank, then*

$$(*) \qquad \mathrm{rk}(\mathsf{K}) \geq \dim_{\mathbb{Q}}(\mathrm{ord}(\mathsf{K})).$$

Actually, smoothness seems mainly a technical condition that can be verified in some concrete cases, and is then further only used via $(*)$.

Let us say $\tilde{T}$ is of **rational type** if $\tilde{T}$ is O-minimal and every model K of finite rank satisfies $(*)$. We introduce this notion here because there may be ways of verifying $(*)$ other than via smoothness. (For example, RCF satisfies $(*)$ by simple valuation theory, and this is essential in Wilkie's proof of "smooth" $\Rightarrow$ "rational type".) We have the following result derived by Wilkie [W2] for smooth $\tilde{T}$. His proof goes through unchanged for $\tilde{T}$ of rational type.

Proposition: *Suppose $\tilde{T}$ is of rational type, $\mathsf{K} \models \tilde{T}$ and k is an elementary submodel of K such that $\mathrm{rk}(\mathsf{K}|\mathsf{k})$ is finite. Then $\mathrm{rk}(\mathsf{K}|\mathsf{k}) \geq \dim_{\mathbb{Q}}(\mathrm{ord}(\mathsf{K})/\mathrm{ord}(\mathsf{k}))$.*

2. $\mathbb{R}_{\mathrm{an}}$ is smooth and $\mathbb{R}\{X,m\}$ is noetherian

2.1 For each $f \in \mathbb{R}\{X,m\}$, define the real analytic function $\hat{f}\colon \mathbb{R}^m \to \mathbb{R}$ by

$$\hat{f}(x_1,\dots,x_m) := f((1+x_1^2)^{-1},\dots,(1+x_m^2)^{-1}),$$

and put

$$\hat{\mathbb{R}}_{\mathrm{an}} := (\mathbb{R},<,0,1,-,+,\cdot,(\hat{f})_{f\in\mathbb{R}\{X,m\},m\in\mathbb{N}}).$$

One checks easily that $\hat{\mathbb{R}}_{\mathrm{an}}$ is interdefinable, both existentially and universally, with $\mathbb{R}_{\mathrm{an}}$, so model completeness and O-minimality of $\mathbb{R}_{\mathrm{an}}$ implies that $\hat{\mathbb{R}}_{\mathrm{an}}$ is model complete and O-minimal. We now use this interdefinability the other way around:

2.2 Lemma: *$\mathbb{R}_{\mathrm{an}}$ is smooth.*

Proof: For the O-minimality condition (S1), see [vdD2], and for the polynomial growth condition (S2), see [vdD2] and [W1]. It remains to show that $\mathbb{R}_{\mathrm{an}}$ satisfies (S3), and this is done along the lines of the argument in [W2] for the structure $\mathbb{R}_e := (\mathbb{R},<,0,1,+,-,\cdot,e)$, where $e(x)=\exp((1+x^2)^{-1})$: Work with $\hat{\mathbb{R}}_{\mathrm{an}}$ instead of $\mathbb{R}_{\mathrm{an}}$, use that $\hat{\mathbb{R}}_{\mathrm{an}}$ is model complete, and that the functions $\hat{f}$ are C^∞ on their domain of definition $\mathbb{R}^m$. Introduce for each $f \in \mathbb{R}\{X,m\}$ and subset s of $\{1,\dots,m\}$ the C^∞-function $f_s\colon \mathbb{R}^m \to \mathbb{R}$ by

$$f_s(x_1,\dots,x_m) := f(x_1',\dots,x_m'),$$

where $x_i' = (1+x_i^2)^{-1}$ for $i \notin s$ and $x_i' = x_i^2(1+x_i^2)^{-1}$ for $i \in s$. Note that then f_s is also definable in $\mathbb{R}_{\mathrm{an}}$, and that if $x_1,\dots,x_m \in \mathbb{R}$ with $x_i \neq 0$ for $i \in s$, then $\hat{f}(y_1,\dots,y_m) = f_s(x_1,\dots,x_m)$, where $y_i = x_i$ for $i \in s$, and $y_i = x_i^{-1}$ for $i \in s$. ∎

2.3 For later use we also note that the ring $\mathbb{R}\{X,m\}$ is noetherian. This follows from the theorem of Frisch [F] that the ring $\mathbb{C}\{X,m\}$ consisting of all $f \in \mathbb{C}[\![X_1,\dots,X_m]\!]$ such that f converges on a neighborhood of the closed polydisc in $\mathbb{C}^m$ with center $(0,\dots,0)$ and polyradius $(1,\dots,1)$ is noetherian, and the fact that $\mathbb{C}\{X,m\} = \mathbb{R}\{X,m\} \oplus i\cdot\mathbb{R}\{X,m\}$.

Let $\hat{\mathbb{R}}\{X,m\}$ be the (noetherian) ring of all functions $\hat{f}$ for $f \in \mathbb{R}\{X,m\}$. For $m \geq 1$ the ring $\hat{\mathbb{R}}\{X,m\}$ does not contain the coordinate functions x_i, nor is $\hat{\mathbb{R}}\{X,m\}$ closed under the operators $\partial/\partial x_i$, and so we adjoin the x_i's and note that $\hat{\mathbb{R}}\{X,m\}[x_1,\dots,x_m]$ is a noetherian ring of real analytic functions on $\mathbb{R}^m$, closed under the operators $\partial/\partial x_i$.

3. Systems of C^∞-rings and Hovanskii's Theorem

3.1 DEFINITION. A **system of C^∞-rings** is a sequence $\mathfrak{R} = (\mathfrak{R}_m)_{m\in\mathbb{N}}$ such that for each m:

(1) $\mathfrak{R}_m$ is a ring of C^∞-functions $f\colon \mathbb{R}^m \to \mathbb{R}$ under pointwise addition and multiplication of functions;
(2) the coordinate functions $x_i\colon \mathbb{R}^m \to \mathbb{R}$ belong to $\mathfrak{R}_m$;
(3) for $f \in \mathfrak{R}_m$, the function $(x_1,\dots,x_m,x_{m+1}) \mapsto f(x_1,\dots,x_m)\colon \mathbb{R}^{m+1} \to \mathbb{R}$ belongs to $\mathfrak{R}_{m+1}$, and for each permutation s of $\{1,\dots,m\}$ the function $(x_1,x_2,\dots,x_m) \mapsto f(x_{s(1)},\dots,x_{s(m)})\colon \mathbb{R}^m \to \mathbb{R}$ belongs to $\mathfrak{R}_m$;
(4) $f \in \mathfrak{R}_m \Rightarrow \partial f/\partial x_i \in \mathfrak{R}_m$, for $i = 1,\dots,m$.

(Of course, in (3) it suffices to consider just permutations $s = (i, i+1)$ with $1 \le i < m$.)

To abbreviate, we just write "system" instead of "system of C^∞-rings".

3.2 EXAMPLES.

1. $(\mathbb{Z}[x_1,\dots,x_m])_{m\in\mathbb{N}}$ is a system, and in an obvious sense the smallest possible system;
2. $(\mathbb{R}[x_1,\dots,x_m])_{m\in\mathbb{N}}$ is a system;
3. $(\hat{\mathbb{R}}\{X,m\}[x_1,\dots,x_m])_{m\in\mathbb{N}}$ is a system (see (2.3));
4. $(\mathbb{Z}[x_1,\dots,x_m,\exp(x_1),\dots,\exp(x_m)])_{m\in\mathbb{N}}$ is a system;
5. $(\mathbb{R}[x_1,\dots,x_m,\exp(x_1),\dots,\exp(x_m)])_{m\in\mathbb{N}}$ is a system.

3.3 We say that the system $\mathfrak{R}$ has the **Hovanskii property** (or, more briefly, that $\mathfrak{R}$ is an ***H*-system**) if for all $m \in \mathbb{N}$ and $f_1,\dots,f_m \in \mathfrak{R}_m$ there is a bound $H = H(f_1,\dots,f_m) \in \mathbb{N}$ such that for all $a = (a_1,\dots,a_m) \in \mathfrak{R}_m$ we have

$$(*) \qquad \operatorname{card}\{x \in \mathbb{R}^m\colon f_1(x) = a_1,\dots,f_m(x) = a_m, J(f_1,\dots,f_m)(x) \neq 0\} \le H.$$

To formulate this geometrically, consider a C^∞-map $f = (f_1,\dots,f_m)\colon \mathbb{R}^m \to \mathbb{R}^m$ and $a \in \mathbb{R}^m$. Then $(*)$ above says that $\operatorname{card}(\operatorname{Reg}(f,a)) \le H$.

If $\operatorname{Reg}(f,a)$ is finite then by the inverse function theorem there is a neighborhood U of a such that $\operatorname{card}(\operatorname{Reg}(f,a')) \ge \operatorname{card}(\operatorname{Reg}(f,a))$ for all a' in U. Now by Sard's lemma such a neighborhood always contains regular values of f, hence in the definition of "Hovanskii property" it suffices to require the existence of the bound H just for regular values a of the map $f = (f_1, \ldots, f_m)$, that is, for a in $\mathbb{R}^m$ with $\mathrm{Reg}(f, a) = f^{-1}(a)$.

The 5 examples above have the Hovanskii property. For the first two this follows from Bezout's theorem, or alternatively, one can apply the following simple observation to the ordered field of reals:

Observation: If $\mathfrak{R}$ is a system such that all functions from $\mathfrak{R}$ are definable in a fixed O-minimal expansion $\tilde{\mathbb{R}}$ of the ordered field of reals, then $\mathfrak{R}$ is an H-system. (See end of (1.2).) ∎

That the third example has the H-property follows from this observation by noting that all functions of $\hat{\mathbb{R}}\{X, m\}[x_1, \ldots, x_m]$ are definable in the O-minimal structure $\mathbb{R}_{\mathrm{an}}$.

That the last two examples have the H-property follows from Hovanskii's theorem 1 in [H]. We will extend Hovanskii's theorem to a method for adjoining new functions to an H-system to get a larger H-system.

3.4 Let $\mathfrak{R}$ be a system. We say that the C^∞-functions $f_1, \ldots, f_k \colon \mathbb{R}^n \to \mathbb{R}$ form a **Pfaffian chain on $\mathbb{R}^n$ over $\mathfrak{R}$** (or a $(P, \mathfrak{R})$-chain on $\mathbb{R}^n$) if for each $j = 1, \ldots, k$ there are functions $p_{ij} \in \mathfrak{R}_{n+j}$, for $i = 1, \ldots, n$, such that

$$(\partial f_j/\partial x_i)(x) = p_{ij}(x, f_1(x), \ldots, f_j(x)) \quad \text{on } \mathbb{R}^n.$$

We call k the **length** of the chain.

3.5 Let $\mathfrak{R}$ be a system and let C^∞-functions $f_1, \ldots, f_k \colon \mathbb{R}^n \to \mathbb{R}$ be given, not necessarily in $\mathfrak{R}_n$. We let $\mathfrak{R}_n\langle f_1, \ldots, f_k\rangle$ consist of all functions of the form

$$x \mapsto p(x, f_1(x), \ldots, f_k(x)) \colon \mathbb{R}^n \to \mathbb{R}, \quad \text{where } p \in \mathfrak{R}_{n+k}.$$

So $\mathfrak{R}_n\langle f_1, \ldots, f_k\rangle$ is a ring of C^∞-functions on $\mathbb{R}^n$, and is the image of $\mathfrak{R}_{n+k}$ under the ring homomorphism $p \mapsto (x \mapsto p(x, f_1(x), \ldots, f_k(x)))$ from $\mathfrak{R}_{n+k}$ into the ring of all C^∞-functions on $\mathbb{R}^n$. In particular, if $\mathfrak{R}_{n+k}$ is noetherian, so is $\mathfrak{R}_n\langle f_1, \ldots, f_k\rangle$. Note that $f_1, \ldots, f_k$ is a $(P, \mathfrak{R})$-chain precisely when the partials $\partial f_j/\partial x_i$ belong to $\mathfrak{R}_n\langle f_1, \ldots, f_j\rangle$, for each $j = 1, \ldots, k$, and that in that case $\mathfrak{R}_n\langle f_1, \ldots, f_k\rangle$ is closed under the operators $\partial/\partial x_i$.

3.6 PROPOSITION: *Let $\mathfrak{R}$ be an H-system, and $f_1, \ldots, f_k$ a $(P, \mathfrak{R})$-chain on $\mathbb{R}^n$ of length k. Then, given any n functions $g_1, \ldots, g_n$ in $\mathfrak{R}_n\langle f_1, \ldots, f_k\rangle$ defining the map $g = (g_1, \ldots, g_n)$: $\mathbb{R}^n \to \mathbb{R}^n$ there is an integer M such that*

$$\operatorname{card}(\operatorname{Reg}(g, a)) \leq M \quad \textit{for all } a \in \mathbb{R}^n.$$

Remark: This is proved just like theorem 1 in [H], with minor modifications. For the reader's convenience we repeat the argument.

Proof: By induction on k. For $k = 0$ this is just the hypothesis that $\mathfrak{R}$ is an H-system. Let $k > 0$, and let $g = (g_1, \ldots, g_n)$: $\mathbb{R}^n \to \mathbb{R}^n$ with $g_i \in \mathfrak{R}_n\langle f_1, \ldots, f_k\rangle$. Write $g_i(x) = h_i(x, f_1(x), \ldots, f_k(x))$ with $h_i \in \mathfrak{R}_{n+k}$. Given $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ we replace the system of equations $g_1(x) = a_1, \ldots, g_n(x) = a_n$ by the equivalent system $F_1(x, v) = a_1, \ldots\ldots, F_n(x, v) = a_n$, $G(x, v) = 0$, where $F_1, \ldots, F_n, G$: $\mathbb{R}^{n+1} \to \mathbb{R}$ are defined by $F_i(x, v) = h_i(x, f_1(x), \ldots, f_{k-1}(x), v)$, $G(x, v) = f_k(x) - v$. Note that $F_1, \ldots, F_n \in \mathfrak{R}_{n+1}\langle f_1, \ldots, f_{k-1}\rangle$; here $f_1, \ldots, f_{k-1}$ are considered as functions on $\mathbb{R}^{n+1}$ that do not depend on the last variable v. Put $F = (F_1, \ldots, F_n)$: $\mathbb{R}^{n+1} \to \mathbb{R}^n$ and note that $x \mapsto (x, f_k(x))$: $\mathbb{R}^n \to \mathbb{R}^{n+1}$ maps $g^{-1}(a)$ bijectively onto $(F, G)^{-1}(a, 0)$, and $\operatorname{Reg}(g, a)$ onto $\operatorname{Reg}((F, G), (a, 0))$. Hence,

$$\operatorname{card}(\operatorname{Reg}(g, a)) = \operatorname{card}(\operatorname{Reg}((F, G), (a, 0)).$$

The partial derivatives of the F_i's belong to $\mathfrak{R}_{n+1}\langle f_1, \ldots, f_{k-1}\rangle$, and on the hypersurface $G(x, v) = 0$ in $\mathbb{R}^{n+1}$ the partial derivatives of G are also given by functions in $\mathfrak{R}_{n+1}\langle f_1, \ldots, f_{k-1}\rangle$, since on this surface we have $f_k(x) = v$. Therefore there is a function $\hat{J} \in \mathfrak{R}_{n+1}\langle f_1, \ldots, f_{k-1}\rangle$ such that on this hypersurface we have $\hat{J}(x, v) =$ the Jacobian determinant of (F, G) at (x, v). By the inductive hypothesis there is a bound $N \in \mathbb{N}$ such that $\operatorname{card}(\operatorname{Reg}((F, \hat{J}), (a, r))) \leq N$ for all $(a, r) \in \mathbb{R}^{n+1}$. In this situation theorem $2'$ from [H] implies: if $q \in \mathbb{N}$ is a bound for the number of non-compact connected components of the curve $F^{-1}(a)$, for every regular value a of F, then $\operatorname{card}(\operatorname{Reg}((F, G), (a, 0)) \leq N + q$ for all $a \in \mathbb{R}^n$. So we are done if we establish the existence of the bound q. For this, one uses again the inductive hypothesis and the fact that the number of non-compact connected components of a curve $F^{-1}(a)$ ($a \in \mathbb{R}^n$ a regular value of F) is at most the maximal number of transversal intersections of $F^{-1}(a)$ with hyperplanes in $\mathbb{R}^{n+1}$. ∎

We will not actually use this proposition until section 5, but it helps to motivate the following step towards proving model completeness of certain expansions of the ordered field of reals, which is due to Wilkie for expansions by Pfaffian functions. We adapt it here to our context, which includes the case $(\mathbb{R}_{\mathrm{an}}, \exp)$.

3.7 Let $\mathfrak{R}$ be a system. Let $L(\mathfrak{R})$ be the language $\{<, 0, 1, +, -, \cdot\}$ of ordered rings augmented by an n-ary function symbol f for each $f \in \mathfrak{R}_n$, and each $n \in \mathbb{N}$. We interpret this symbol f as the corresponding function f on $\mathbb{R}^n$ and obtain in this way an expansion of the ordered field of real numbers whose complete theory we denote by $T_{\mathfrak{R}}$. If we further extend $L(\mathfrak{R})$ by a unary function symbol exp to get the language $L(\mathfrak{R}, \exp)$, and interpret this symbol as the usual exponential function on $\mathbb{R}$, then we denote the complete $L(\mathfrak{R}, \exp)$-theory of the corresponding expansion of the ordered field of reals by $T_{\mathfrak{R},\exp}$. Note that exp is a $(P, \mathfrak{R})$-chain of length 1 on $\mathbb{R}$. More generally, given any functions $f_1, \ldots, f_k \colon \mathbb{R}^n \to \mathbb{R}$, we extend $L(\mathfrak{R})$ by new n-ary function symbols $f_1, \ldots, f_k$ to get the language $L(\mathfrak{R}, f_1, \ldots, f_k)$, and the complete theory of the resulting $L(\mathfrak{R}, f_1, \ldots, f_k)$-expansion of $\mathbb{R}$ is denoted by $T_{\mathfrak{R}, f_1, \ldots, f_k}$. Given a model K of $T_{\mathfrak{R}, f_1, \ldots, f_k}$ we let $L(\mathfrak{R}, f_1, \ldots, f_k, \mathsf{K})$ be the language $L(\mathfrak{R}, f_1, \ldots, f_k)$ augmented by constants for the elements of K.

Let $\mathfrak{R}_n[\exp(x_1), \ldots, \exp(x_n)]$ be the subring of the ring of C^∞-functions on $\mathbb{R}^n$ generated by $\exp(x_1), \ldots, \exp(x_n)$ over $\mathfrak{R}_n$. Note that $\mathfrak{R}_n[\exp(x_1), \ldots, \exp(x_n)]$ is closed under the operators $\partial/\partial x_i$.

Let $\mathsf{K} \models T_{\mathfrak{R},\exp}$.

By writing each $f \in \mathfrak{R}_n[\exp(x_1), \ldots, \exp(x_n)]$ as a polynomial in the $\exp(x_i)$ over $\mathfrak{R}_n$ we can associate to each such f a definable C^∞-function $f_{\mathsf{K}} \colon \mathsf{K}^n \to \mathsf{K}$, and this function f_{K} does not depend on the particular way of representing f as a polynomial in the $\exp(x_i)$ over $\mathfrak{R}_n$; the map $f \mapsto f_{\mathsf{K}}$ is an injective ring homomorphism from $\mathfrak{R}_n[\exp(x_1), \ldots, \exp(x_n)]$ into the ring of all C^∞-functions on K^n, and this map satisfies also $(\partial f/\partial x_i)_{\mathsf{K}} = \partial f_{\mathsf{K}}/\partial x_i$. Hence there is no harm in leaving out the subscript K in f_{K} and considering $\mathfrak{R}_n[\exp(x_1), \ldots, \exp(x_n)]$ also as a ring of functions on K^n. Given $c_1, \ldots, c_r \in \mathsf{K}$, we denote by

$$\mathfrak{R}_n[\exp(x_1), \ldots, \exp(x_n), c_1, \ldots, c_r]$$

the ring of C^∞-functions on K^n generated by the constant functions $c_1, \ldots, c_r$ over $\mathfrak{R}_n[\exp(x_1), \ldots, \exp(x_n)]$. Note that $\mathfrak{R}_n[\exp(x_1), \ldots, \exp(x_n), c_1, \ldots, c_r]$ is then also closed under the operations $\partial/\partial x_i$, and is noetherian if $\mathfrak{R}_n$ is.

Call $\mathfrak{R}$ **noetherian** if each ring $\mathfrak{R}_n$ is noetherian. (The five examples at the beginning of this section are all noetherian.)

We now have the following variant of lemma 2.7 in [W1]:

3.8 LEMMA: *Let $\mathfrak{R}$ be noetherian and* k, $\mathsf{K} \models T_{\mathfrak{R},\exp}$ *with* $\mathsf{k} \subseteq \mathsf{K}$. *Let*

$$t_1(x_1,\dots,x_m),\dots,t_k(x_1,\dots,x_m)$$

be $L(\mathfrak{R},\exp,\mathsf{k})$*-terms such that the system*

$$(*) \qquad t_1(x_1,\dots,x_m)=\dots=t_k(x_1,\dots,x_m)=0$$

has a solution in K^m. *Then there exist* $f_1,\dots,f_n \in \mathfrak{R}_n[\exp(x_1),\dots,\exp(x_n)]$ *and* $b_1,\dots,b_n \in \mathsf{k}$ *for some* $n \geq m$ *such that the system of equations*

$$\begin{gathered} f_1(x_1,\dots,x_n)=b_1 \\ \vdots \\ f_n(x_1,\dots,x_n)=b_n \end{gathered}$$

has a regular solution $(a_1,\dots,a_n)\in \mathsf{K}^n$ *with* $(a_1,\dots,a_m)$ *a solution of* $(*)$.

Proof: If a term $t_i(x_1,\dots,x_m)$ is of the form $t(x_1,\dots,x_m,\tau(x_1,\dots,x_m))$ for "simpler" $L(\mathfrak{R},\exp,\mathsf{k})$-terms $t(x_1,\dots,x_m,x_{m+1})$ and $\tau(x_1,\dots,x_m)$, we can replace the equation $t_i(x_1,\dots,x_m)=0$ in $(*)$ by the two equations

$$\begin{gathered} t(x_1,\dots,x_m,x_{m+1})=0 \\ \tau(x_1,\dots,x_m)-x_{m+1}=0 \end{gathered}$$

By unravelling the terms t_i in this way and increasing m we may reduce to the case that all t_i define functions on K^m from $\mathfrak{R}_m[\exp(x_1),\dots,\exp(x_m),c_1,\dots,c_r]$ for certain $c_1,\dots,c_r \in \mathsf{k}$. Since this ring is noetherian and closed under the operations $\partial/\partial x_i$ we can apply Theorem 5.1 from [W1] and conclude there are

$$g_1,\dots,g_m \in \mathfrak{R}_m[\exp(x_1),\dots,\exp(x_m),c_1,\dots,c_r]$$

such that $V(t_1,\dots,t_k)\cap V^{\mathrm{reg}}(g_1,\dots,g_m)$ is non-empty. By replacing $c_1,\dots,c_r$ in the terms representing the g's by new variables $x_{m+1},\dots,x_{m+r}$ we obtain functions $f_1,\dots,f_m \in \mathfrak{R}_n[\exp(x_1),\dots,\exp(x_n)]$ on K^n for $n=m+r$. Put $f_{m+1}:=x_{m+1},\dots,f_{m+r}:=x_{m+r}$, $b_1=\dots=b_m:=0$ and $b_{m+1}:=c_1,\dots,b_{m+r}:=c_r$, and we have functions $f_1,\dots,f_n$ and elements $b_1,\dots,b_n$ with the desired property. ∎

This lemma has (by Robinson's test) the following consequence.

3.9 Lemma: *Suppose $\mathfrak{R}$ is noetherian. Then $T_{\mathfrak{R},\exp}$ is model complete if for all models* k *and* K *of* $T_{\mathfrak{R},\exp}$ *with* $\mathsf{k} \subseteq \mathsf{K}$ *and every n functions* $f_1, \ldots, f_n \in \mathfrak{R}_n[\exp(x_1), \ldots, \exp(x_n)]$ *and every n elements* $b_1, \ldots, b_n \in \mathsf{k}$, *every regular solution in* K^n *of the system of equations*

$$\begin{gathered} f_1(x_1, \ldots, x_n) = b_1 \\ \vdots \\ f_n(x_1, \ldots, x_n) = b_n \end{gathered}$$

belongs to k^n.

This lemma reduces the problem of proving model completeness of $T_{\mathfrak{R},\exp}$ to more managable proportions.

4. Bounding Regular Solutions

4.1 An **ordered exponential field** is here a pair $(\mathsf{K}, \exp)$ with K an ordered field and $\exp\colon \mathsf{K} \to \mathrm{Pos}(\mathsf{K})$ a strictly increasing isomorphism from the additive group of K onto the multiplicative group $\mathrm{Pos}(\mathsf{K})$, such that in addition $\exp(1) \in \mathrm{Fin}(\mathsf{K})$ and there is for each $n \in \mathbb{N}$ an $E(n) \in \mathbb{N}$ such that $\exp(x) > x^n$ for all $x > E(n)$. When dealing with $\mathbb{R}$ the symbol exp will always denote the usual exponential function $x \mapsto e^x$, but note that for any real number $a > 1$ the function $x \mapsto a^x$ makes $\mathbb{R}$ into an ordered exponential field in the sense just defined.

For the rest of this section we fix an ordered exponential field $(\mathsf{K}, \exp)$.

Observe that $\mathrm{Pos}(\mathsf{K})$ is a divisible group, hence $\mathrm{ord}(\mathsf{K})$ is a divisible group. We may therefore consider $\mathrm{ord}(\mathsf{K})$ as a $\mathbb{Q}$-linear space. Note also that the map $x \mapsto \mathrm{ord}(\exp(x))\colon \mathsf{K} \to \mathrm{ord}(\mathsf{K})$ is $\mathbb{Q}$-linear with kernel $\mathrm{Fin}(\mathsf{K})$.

4.2 We now give an algebraic version of a lemma due to Wilkie [W2]. Consider subfields k and k^* of K such that

(i) $\mathsf{k} \subseteq \mathsf{k}^*$ and $\mathrm{Pos}(\mathsf{k}^*)$ is a divisible group,
(ii) $\exp(\mathsf{k}) = \mathrm{Pos}(\mathsf{k})$,
(iii) $\exp(x) \in \mathsf{k}^*$ for all $x \in \mathsf{k}^*$ with $|x| < 1$.

For each $a \in \mathsf{k}^*$, $\exp(a)$ is an element of K that may or may not lie in k^*. Put $L(\mathsf{k}^*) := \{a \in \mathsf{k}^*\colon \exp(a) \in \mathsf{k}^*\}$, a $\mathbb{Q}$-linear subspace of k^* containing k and $\mathrm{Fin}(\mathsf{k}^*)$. Also $\mathrm{ord}(\mathsf{k})$ and $\mathrm{ord}(\mathsf{k}^*)$ are $\mathbb{Q}$-linear spaces.

4.3 Lemma:

(i) *The* $\mathbb{Q}$*-linear map* $a \mapsto \operatorname{ord}(\exp(a))\colon L(\mathrm{k}^*) \to \operatorname{ord}(\mathrm{k}^*)$ *has kernel* $\operatorname{Fin}(\mathrm{k}^*)$ *and induces on residue classes an injective* $\mathbb{Q}$*-linear map*

$$L(\mathrm{k}^*)/(\mathrm{k} + \operatorname{Fin}(\mathrm{k}^*)) \to \operatorname{ord}(\mathrm{k}^*)/\operatorname{ord}(\mathrm{k}).$$

In particular, $\dim_{\mathbb{Q}}(L(\mathrm{k}^*)/(\mathrm{k}+\operatorname{Fin}(\mathrm{k}^*))) \leq \dim_{\mathbb{Q}}(\operatorname{ord}(\mathrm{k}^*)/\operatorname{ord}(\mathrm{k}))$.

(ii) *If* $\dim_{\mathbb{Q}}(\operatorname{ord}(\mathrm{k}^*)/\operatorname{ord}(\mathrm{k})) < \infty$ *and* $L(\mathrm{k}^*)$ *contains an element that is not* k*-bounded, then* $\dim_{\mathbb{Q}}(L(\mathrm{k}^*)/(\mathrm{k}+\operatorname{Fin}(\mathrm{k}^*))) < \dim_{\mathbb{Q}}(\operatorname{ord}(\mathrm{k}^*)/\operatorname{ord}(\mathrm{k}))$.

Proof (Identical to the proof of lemma 4.2 in [W2]): Let $U := \mathrm{k} + \operatorname{Fin}(\mathrm{k}^*)$. To prove (i), let $a \in L(\mathrm{k}^*)$ and assume that $\operatorname{ord}(\exp(a)) \in \operatorname{ord}(\mathrm{k})$. Hence there is non-zero $c \in \mathrm{k}$ such that $\operatorname{ord}(\exp(a)) = \operatorname{ord}(c)$. We may assume $c > 0$, and take $d \in \mathrm{k}$ with $\exp(d) = c^{-1}$, so that $N^{-1} < \exp(d+a) < N$ for some positive integer N, hence $d + a \in \operatorname{Fin}(\mathrm{k}^*)$, so $a \in \mathrm{k} + \operatorname{Fin}(\mathrm{k}^*)$.

For (ii), assume $m := \dim_{\mathbb{Q}}(\operatorname{ord}(\mathrm{k}^*)/\operatorname{ord}(\mathrm{k})) < \infty$, that $L(\mathrm{k}^*)$ contains an element that is not k-bounded, and that $\dim_{\mathbb{Q}}(L(\mathrm{k}^*)/U) = m$. We shall derive a contradiction. Choose positive elements $a_1 < \cdots < a_m$ in $L(\mathrm{k}^*)$ such that $a_1 + U, \ldots, a_m + U$ is a basis of $L(\mathrm{k}^*)/U$, and such that for each $v \in L(\mathrm{k}^*)$ with $v > U$, if $v = q_1a_1 + \cdots + q_ma_m + u$, ($q_i \in \mathbb{Q}$, $u \in U$), then $v > qa_j$ for some positive rational q, where $j = \max\{i\colon q_i \neq 0\}$. Note that $\exp(a_1), \ldots, \exp(a_m) \in \mathrm{k}^*$. By part (i) the elements $\operatorname{ord}(\exp(a_1)), \ldots, \operatorname{ord}(\exp(a_m))$ span $\operatorname{ord}(\mathrm{k}^*)$ over $\operatorname{ord}(\mathrm{k})$, so if j is minimal with the property that $a_j > \mathrm{k}$, there are $q_1, \ldots, q_m \in \mathbb{Q}$ such that $\operatorname{ord}(\exp(q_1a_1 + \cdots + q_ma_m)) = \operatorname{ord}(a_j) + \operatorname{ord}(c)$ for some non-zero $c \in \mathrm{k}$. As before this gives $d \in \mathrm{k}$ with $\operatorname{ord}(\exp(d + q_1a_1 + \cdots + q_ma_m)) = \operatorname{ord}(a_j)$, so $a_j/N < \exp(d + q_1a_1 + \cdots + q_ma_m) < Na_j$ for some positive integer N. The left hand inequality implies $d + q_1a_1 + \cdots + q_ma_m > \mathrm{k}$, since $a_j > \mathrm{k}$ and k is closed under the monotone increasing function exp. Thus $q_i \neq 0$ for some $i \in \{j, \ldots, m\}$. By the choice of the basis $a_1, \ldots, a_m$ this implies $d + q_1a_1 + \cdots + q_ma_m > qa_j$ for some positive rational q. But then $\exp(d + q_1a_1 + \cdots + q_ma_m) > \exp(qa_j) > Na_j$, contradicting the right hand inequality above. ∎

4.4 Assumptions in proposition (4.5). Let an H-system $\mathfrak{R}$ be given such that

(i) $\mathfrak{R}_1$ contains the function $x \mapsto (1 + x^2)^{-1}\colon \mathbb{R} \to \mathbb{R}$, as well as the function $e\colon \mathbb{R} \to \mathbb{R}$, where $e(x) = \exp((1 + x^2)^{-1})$;

(ii) $T_{\mathfrak{R}}$ is of rational type (for example smooth).

Let also K have extra structure so that $\mathsf{K} \models T_{\mathfrak{R}}$; in particular the function $e \in \mathfrak{R}_1$ has an interpretation $e_{\mathsf{K}}: \mathsf{K} \to \mathsf{K}$, and more generally each function $g \in \mathfrak{R}_N$ has an interpretation as a C^∞-function $g_{\mathsf{K}}: \mathsf{K}^N \to \mathsf{K}$. We assume further that the exponential map exp on K satisfies $\exp((1+x^2)^{-1}) = e_{\mathsf{K}}(x)$ for all $x \in \mathsf{K}$. (These condition are all satisfied if $(\mathsf{K}, \exp) \models T_{\mathfrak{R},\exp}$.)

Note that then exp is a C^∞-function on K with $\exp'(x) = \exp(x)$ for all x. In the following we shall drop the subscript K in g_{K} for g in $\mathfrak{R}$.

We finally assume k is an elementary substructure of the $T_{\mathfrak{R}}$-model K such that $\exp(\mathsf{k}) = \mathrm{Pos}(\mathsf{k})$. Under these assumptions we have:

4.5 PROPOSITION: *Let $t_1(x,y), \ldots, t_n(x,y)$ be $L(\mathfrak{R}, \mathsf{k})$-terms, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$, $m \leq n$, and let $a = (a_1, \ldots, a_n) \in \mathsf{K}^n$ be a regular solution of the system of C^∞-equations*

$$\begin{gathered} t_1(x, \exp(x_1), \ldots, \exp(x_m)) = 0 \\ \vdots \\ t_n(x, \exp(x_1), \ldots, \exp(x_m)) = 0 \end{gathered}$$

Then $a_1, \ldots, a_m$ are k-bounded.

Remark: Only the case $m = n$ is relevant, but the particular induction in the proof suggests the formulation with $m \leq n$.

Proof: By induction on m. For $m = 0$ there is nothing to prove. Assume $m > 0$ and put $a_{n+j} := \exp(a_j)$ for $j = 1, \ldots, m$ and $\tilde{a} := (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m})$, $\tilde{x} := (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$, so $\tilde{a}$ is a regular solution of the system

$$\begin{array}{ll} t_i(\tilde{x}) = 0 & (i = 1, \ldots, n) \\ \exp(x_j) - x_{n+j} = 0 & (j = 1, \ldots, m). \end{array}$$

(From now on the index i runs over $\{1, \ldots, n\}$ and j over $\{1, \ldots, m\}$.)

Take $S \subseteq \{1, \ldots, n+m\}$ with $\mathrm{card}(S) = n$ such that the n-by-n matrix

$$((\partial t_i / \partial x_s)(\tilde{a}))_{1 \leq i \leq n,\ s \in S}$$

is nonsingular. Let k^* be the elementary substructure of the $T_{\mathfrak{R}}$-model K generated over k by the a_s for $s \in \{1, \ldots, n+m\} \backslash S$. Then $(a_s)_{s \in S}$ is a regular solution of the system

$$\rho_i(x') = 0 \quad (i = 1, \ldots, n)$$

where $x' = (x_s)_{s\in S}$ and $\rho_i(x')$ is the $L(\mathfrak{R}, \mathrm{k}^*)$-term obtained by substituting a_s for x_s in $t_i(\tilde{x})$ for $s \in \{1, \dots, n+m\}\backslash S$. Since $T_{\mathfrak{R}}$ is O-minimal the ρ-system can have only finitely many regular solutions in K^n, and because k^* is an elementary substructure of K this implies $a_s \in \mathrm{k}^*$ for $s \in S$ as well. Suppose some a_j with $1 \le j \le m$ is not k-bounded, say $|a_1| > \mathrm{k}$. We will derive a contradiction. Since $\exp(a_1) = a_{n+1}, \dots, \exp(a_m) = a_{n+m}$ lie in k^*, we have $a_1, \dots, a_m \in L(\mathrm{k}^*)$. Hence by the previous lemma and the assumption that $T_{\mathfrak{R}}$ is of rational type we get:

$$a_1, \dots, a_m \text{ are } \mathbb{Q}\text{-linearly dependent over } \mathrm{k} + \mathrm{Fin}(\mathrm{k}^*).$$

So there are integers $k(1), \dots, k(m)$, not all zero, and $c \in \mathrm{k}$, such that

$$c + k(1)a_1 + \cdots + k(m)a_m \in \mathrm{Fin}(\mathrm{k}^*).$$

Since $|a_1| > \mathrm{k}$, some $k(j)$ with $2 \le j \le m$ is nonzero. To simplify notation we may as well assume that $k(m) \neq 0$, $m > 1$. Taking negatives and changing c if necessary we may even assume:

$$k(m) > 0 \text{ and } 0 < c + k(1)a_1 + \cdots + k(m)a_m < 1.$$

Take $d \in \mathrm{k}^*$ such that $c + \sum k(j)a_j = (1+d^2)^{-1}$. Note that $d \neq 0$. Exponentiating this relation and rearranging gives:

$$\exp(c) \cdot (\prod_{j<m} \exp(a_j)^{k(j)}) \cdot \exp(a_m)^{k(m)} - e(d) = 0.$$

Now consider the following system $H(x_1, \dots, x_n, x_{n+1}, x_{n+2})$ of $(n+2)$ equations over k:

$$t_i(x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_{m-1}), x_{n+1}) = 0 \quad (1 \le i \le n)$$
$$(c + \sum k(j)x_j) - (1 + x_{n+2}^2)^{-1} = 0$$
$$\exp(c) \cdot (\prod_{j<m} \exp(x_j)^{k(j)}) \cdot x_{n+1}^{k(m)} - e(x_{n+2}) = 0.$$

Clearly $(a_1, \dots, a_n, \exp(a_m), d)$ is a solution of this system. Let $h_1, \dots, h_n$, h_{n+1}, h_{n+2}: $\mathrm{K}^{n+2} \to \mathrm{K}$ be the definable C^∞-functions on the left hand side of this system H, in particular, for $(x_1, \dots, x_{n+2}) \in \mathrm{K}^{n+2}$ and $1 \le i \le n$:

$$h_i(x_1, \dots, x_n, \exp(x_m), x_{n+2}) = t_i(x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_m)),$$
$$h_{n+1}(x_1, \dots, x_{n+2}) = (c + \sum k(j)x_j) - (1 + x_{n+2}^2)^{-1},$$
$$h_{n+2}(x_1, \dots, x_{n+2}) = \exp(c) \cdot (\prod_{j<m} \exp(x_j)^{k(j)}) \cdot x_{n+1}^{k(m)} - e(x_{n+2}).$$

Let $A := \exp(c) \cdot \prod \exp(a_j)^{k(j)}$ and $B := 2d/(1+d^2)^2$, and note that $A = e(d)$. A routine computation using these formulas, row and column operations, and row expansion by the last two rows of the Jacobian matrix shows:

$$J(h_1, \ldots, h_{n+2})(a, \exp(a_m), d) = -k(m) \cdot A \cdot B \cdot \exp(-a_m) \cdot J(p_1, \ldots, p_n)(a),$$

where $a = (a_1, \ldots, a_n)$ and $p_i(x_1, \ldots, x_n) = t_i(x, \exp(x_1), \ldots, \exp(x_m))$. Thus, this Jacobian determinant is non-zero. Hence $(a_1, \ldots, a_n, \exp(a_m), d)$ is a regular solution of the system H, whose equations do not involve $\exp(x_m)$. However, we cannot apply the inductive hypothesis yet, since some $k(j)$ might be negative, in which case h_{n+2} might not be of the required form. But multiplying h_{n+2} by $(\prod_{j<m} \exp(x_j))^p$ for a suitable $p > 0$ we obtain a new equation giving rise to an equivalent system to which we can apply the inductive hypothesis to conclude that $a_1, \ldots, a_{m-1}$ are k-bounded, contradicting the assumption $|a_1| > \mathsf{k}$. This finishes the proof. ∎

4.6 Remarks. (1) Note that the hypotheses in (4.4) are satisfied by the system

$$\mathfrak{R} = (\hat{\mathbb{R}}\{X, m\}[x_1, \ldots, x_m]),$$

and any pair of models $(\mathsf{K}, \exp)$, $(\mathsf{k}, \exp_{\mathsf{k}})$ of $T_{\mathfrak{R},\exp}$ with $(\mathsf{k}, \exp_{\mathsf{k}}) \subsetneq (\mathsf{K}, \exp)$. This is the case that is of interest in connection with proving model completeness of $(\mathbb{R}_{\mathrm{an}}, \exp)$.

(2) Proposition (4.5) corresponds to section 2 in [W2], and most of the proof is along the same lines. However, the hypotheses of (4.5) are weaker than in [W2], since we do not assume that $(\mathsf{K}, \exp)$ and $(\mathsf{k}, \exp|\mathsf{k})$ are models of $T_{\mathfrak{R},\exp}$. The last part of our proof is accordingly different and does not depend on Hovanskii's theorem or results in [W1], unlike the proof in [W2].

5. Substitution, O-Minimality and Finite Model Completeness

In this section we show how to enlarge an H-system by functions of a Pfaffian chain to a "substitution closed" H-system. Under an additional noetherianity assumption on the original system we can then show that zero sets of functions defined by terms have only finitely many connected components, see (5.10). We use this to derive O-minimality from model completeness in (5.11). We also show that under certain conditions model completeness implies finite model completeness; see (5.14).

Let $\mathfrak{R}$ be a system.

5.1 We say that $\mathfrak{R}$ is **substitution closed** if for all $f \in \mathfrak{R}_n$ and $g_1, \ldots, g_n \in \mathfrak{R}_m$ we have $f(g_1, \ldots, g_n) \in \mathfrak{R}_m$. The first two examples of (3.2) are substitution closed, the last three are not.

5.2 Using induction on terms and the chain rule one easily shows that the functions $f\colon \mathbb{R}^m \to \mathbb{R}$ defined by $L(\mathfrak{R})$-terms $t(x_1, \ldots, x_m)$, $(m = 0, 1, 2, \ldots)$, form a system that contains $\mathfrak{R}$ and is substitution closed; it is the smallest such system. We call it the substitution closure of $\mathfrak{R}$ and denote it by $\mathfrak{R}^s$.

5.3 LEMMA: *Let $L(\mathfrak{R})$-terms $t_1(y), \ldots, t_k(y)$ be given, $y = (y_1, \ldots, y_n)$. Then there are $L(\mathfrak{R})$-terms $\tau_1(y), \ldots, \tau_N(y)$ and functions $f_1, \ldots, f_{k+N}$ in $\mathfrak{R}_{n+N}$ such that the map $y \mapsto (y, \tau_1(y), \ldots, \tau_N(y))\colon \mathbb{R}^n \to \mathbb{R}^{n+N}$ maps $V(t_1, \ldots, t_k)$ bijectively onto $V(f_1, \ldots, f_{k+N})$, and $V^{\mathrm{reg}}(t_1, \ldots, t_k)$ onto $V^{\mathrm{reg}}(f_1, \ldots, f_{k+N})$.*

Proof: If all terms $t_i(y)$ define functions in $\mathfrak{R}_n$ we can take $N = 0$ and let f_i be the function defined by $t_i(y)$. If, say, $t_k(y)$ is of the form $t(y, \tau(y))$ with simpler $L(\mathfrak{R})$-terms $t(y, z)$ and $\tau(y)$, z an extra variable, then the system of k equations $t_1(y) = \cdots = t_k(y) = 0$ is equivalent to the system of $k+1$ equations $t_1(y) = \cdots = t_{k-1}(y) = t(y, z) = \tau(y) - z = 0$: the map $y \mapsto (y, \tau(y))\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ maps the solution set of the first system bijectively onto the solution set of the second system, with regular solutions corresponding to regular solutions. Unravelling terms in this way and introducing extra variables we construct in a finite number of steps the desired terms and functions. ∎

5.4 PROPOSITION: *Let $L(\mathfrak{R})$-terms $t_1(x, y), \ldots, t_n(x, y)$ be given, where $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$. Then there are terms $\tau_1(x, y), \ldots, \tau_N(x, y)$ from $L(\mathfrak{R})$ and a map $g = (g_1, \ldots, g_{m+n+N})\colon \mathbb{R}^{m+n+N} \to \mathbb{R}^{m+n+N}$ with all $g_i \in \mathfrak{R}_{m+n+N}$ such that for each $a \in \mathbb{R}^m$ the map*

$$y \mapsto (a, y, \tau_1(a, y), \ldots, \tau_N(a, y))\colon \mathbb{R}^n \to \mathbb{R}^{m+n+N}$$

maps $V(t_1(a, y), \ldots, t_n(a, y))$ bijectively onto $g^{-1}(a, 0, 0)$, and maps

$$V^{\mathrm{reg}}(t_1(a, y), \ldots, t_n(a, y)) \text{ onto } \mathrm{Reg}(g, (a, 0, 0)).$$

Proof: By the lemma there are $L(\mathfrak{R})$-terms $\tau_1(x, y), \ldots, \tau_N(x, y)$ and functions $f_1, \ldots, f_{n+N} \in \mathfrak{R}_{m+n+N}$ such that the map

$$(x, y) \mapsto ((x, y, \tau_1(x, y), \ldots, \tau_N(x, y))\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n+N}$$

maps $V(t_1,\ldots,t_n)$ bijectively onto $V(f_1,\ldots,f_{n+N})$, and $V^{\mathrm{reg}}(t_1,\ldots,t_n)$ onto $V^{\mathrm{reg}}(f_1,\ldots,f_{n+N})$. The way these terms τ_i and functions f_j are obtained in the proof of the lemma also shows that for each fixed $a \in \mathbb{R}^m$ the map $y \mapsto (y,\tau_1(a,y),\ldots,\tau_N(a,y))$: $\mathbb{R}^n \to \mathbb{R}^{n+N}$ maps $V^{\mathrm{reg}}(t_1(a,y),\ldots,t_n(a,y))$ bijectively onto $V^{\mathrm{reg}}(f_1(a,y,z),\ldots,f_{n+N}(a,y,z))$. Thus for fixed $a \in \mathbb{R}^m$ the system of equations $t_1(a,y) = \cdots = t_n(a,y) = 0$ is, in this sense, equivalent to the system $f_1(a,y,z) = \cdots = f_{n+N}(a,y,z) = 0$, which in turn is equivalent to the system $x_1 = a_1,\ldots,x_m = a_m$, $f_1(x,y,z) = \cdots = f_{n+N}(x,y,z) = 0$. Now define $g_i \in \mathfrak{R}_{m+n+N}$ by $g_i = x_i$ for $1 \le i \le m$, $g_{m+j} = f_j$ for $1 \le j \le n+N$, and let $g := (g_1,\ldots,g_{m+n+N})$: $\mathbb{R}^{m+n+N} \to \mathbb{R}^{m+n+N}$. Then one easily verifies the claim in the proposition. ∎

5.5 Corollary: *Suppose $\mathfrak{R}$ is an H-system. Then $\mathfrak{R}^s$ is also an H-system: given any $L(\mathfrak{R})$-terms $t_1(x,y),\ldots,t_n(x,y)$, $x = (x_1,\ldots,x_m)$, $y = (y_1,\ldots,y_n)$, there is a bound $H \in \mathbb{N}$ such that for all $a \in \mathbb{R}^m$ we have*

$$\operatorname{card}(V^{\mathrm{reg}}(t_1(a,y),\ldots,t_n(a,y))) \le H.$$

5.6 Proposition: *Suppose the system $\mathfrak{R}$ is substitution closed and $f_1,\ldots,f_k$ is a $(P,\mathfrak{R})$- chain on $\mathbb{R}^n$. Let $t(x)$ with $x = (x_1,\ldots,x_m)$ be an $L(\mathfrak{R},f_1,\ldots,f_k)$-term, defining the function t: $\mathbb{R}^m \to \mathbb{R}$. Then there is a $(P,\mathfrak{R})$-chain $F_1,\ldots,F_K$ on $\mathbb{R}^m$, such that $t \in \mathfrak{R}_m\langle F_1,\ldots,F_K\rangle$ and each function F_i is defined by an $L(\mathfrak{R},f_1,\ldots,f_k)$-term.*

Proof: By induction on complexity of the term $t(x)$. If $t(x)$ is a constant (given by an element of $\mathfrak{R}_0$) or one of the variables x_i, we can take $K = 0$. If the desired result holds for terms $t_1(x)$ and $t_2(x)$, then also for $t_1(x) + t_2(x)$ and $t_1(x)\cdot t_2(x)$, using the fact that if $F_1,\ldots,F_K$ and $G_1,\ldots,G_L$ are $(P,\mathfrak{R})$-chains on $\mathbb{R}^m$, then $F_1,\ldots,F_K,G_1,\ldots,G_L$ is a $(P,\mathfrak{R})$-chain on $\mathbb{R}^m$. Let now $t(x)$ be $f(\tau_1(x),\ldots,\tau_M(x))$ with $f \in \mathfrak{R}_M$ and $L(\mathfrak{R},f_1,\ldots,f_k)$-terms $\tau_1(x),\ldots,\tau_M(x)$, and assume inductively that we have a $(P,\mathfrak{R})$-chain $F_1,\ldots,F_K$ on $\mathbb{R}^m$ such that $\tau_1,\ldots,\tau_M \in \mathfrak{R}_m\langle F_1,\ldots,F_K\rangle$, where τ_i is the function on $\mathbb{R}^m$ defined by the term $\tau_i(x)$. Take functions $p_1,\ldots,p_M$ in $\mathfrak{R}_{m+K}$ such that $\tau_i(x) = p_i(x,F(x))$ for all $x \in \mathbb{R}^m$ where $F(x) = (F_1(x),\ldots,F_K(x)) \in \mathbb{R}^K$. Then

$$t(x) = f(p_1(x,F(x)),\ldots,p_M(x,F(x))) \text{ for all } x \text{ in } \mathbb{R}^m,$$

and since $\mathfrak{R}$ is substitution closed this gives $t \in \mathfrak{R}_m\langle F_1,\ldots,F_K\rangle$.

Finally, let $t(x)$ be a term $f(\tau_1(x),\dots,\tau_M(x))$ as before, except that instead of $f \in \mathfrak{R}_M$ we assume f is one of the function symbols f_j, $j \in \{1,\dots,k\}$ and $M = n$. We also make the same inductive assumption and use the same notations as in the previous case, so $F_1,\dots,F_K$ and $p_1,\dots,p_n$ and $F(x)$ are as explained above. We also put $\tau(x) := (\tau_1(x),\dots,\tau_n(x)) \in \mathbb{R}^n$ for $x \in \mathbb{R}^m$, and introduce functions $g_1,\dots,g_k$ on $\mathbb{R}^m$ by $g_j(x) := f_j(\tau(x))$, so the function t occurs among $g_1,\dots,g_k$. Hence it suffices to establish the following claim:

CLAIM: *$F_1,\dots,F_K$, $g_1,\dots,g_k$ is a $(P,\mathfrak{R})$-chain on $\mathbb{R}^m$.* To see this, let $h_{rj} \in \mathfrak{R}_{n+j}$ be such that $(\partial f_j/\partial y_r)(y) = h_{rj}(y, f_1(y),\dots,f_j(y))$ on $\mathbb{R}^n$, $j = 1,\dots,k$, $r = 1,\dots,n$, so that

$$\begin{aligned}(\partial g_j/\partial x_i)(x) &= \sum_r (\partial f_j/\partial y_r)(\tau(x)) \cdot (\partial \tau_r/\partial x_i)(x)\\ &= \sum_r h_{rj}(\tau(x), g_1(x),\dots,g_j(x)) \cdot (\partial \tau_r/\partial x_i)(x)\\ &= G_{ij}(x, F(x), g_1(x),\dots,g_j(x)) \text{ on } \mathbb{R}^m,\end{aligned}$$

for some $G_{ij} \in \mathfrak{R}_{m+K+j}$, using the inductive assumption on $\tau_1,\dots,\tau_n$ and the assumption that $\mathfrak{R}$ is substitution closed. ∎

5.7 Let $f_1,\dots,f_k$ be a $(P,\mathfrak{R})$-chain on $\mathbb{R}^n$.

Clearly the functions on $\mathbb{R}^m$, $(m = 0,1,2,\dots)$, defined by $L(\mathfrak{R}, f_1,\dots,f_k)$-terms form a substitution closed system; we denote it by $\mathfrak{R}\langle f_1,\dots,f_k\rangle^s$. From propositions (5.5), (5.6) and (3.6) we deduce:

5.8 COROLLARY: *If $\mathfrak{R}$ is an H-system, so is $\mathfrak{R}\langle f_1,\dots,f_k\rangle^s$.*

Next a differential-topological result whose proof follows Wilkie's elegant proof of a theorem of Hovanskii; cf. Proposition 5.3 from [W1].

5.9 PROPOSITION: *Let $x_1,\dots,x_m,y_1,\dots,y_n$ denote the usual coordinate functions on $\mathbb{R}^{m+n}$, and let R be a noetherian ring of C^∞-functions on $\mathbb{R}^{m+n}$ containing the y_j and closed under the operators $\partial/\partial y_j$ for $j = 1,\dots,n$. Suppose also that for all $f_1,\dots,f_n \in R$ there is a bound $H \in \mathbb{N}$ such that for all $a \in \mathbb{R}^m$ we have $\operatorname{card}(V^{\mathrm{reg}}(f_a)) \le H$, where $f_a\colon \mathbb{R}^n \to \mathbb{R}^n$ is given by*

$$f_a(y) = (f_1(a,y),\dots,f_n(a,y)).$$

Then there is for each function $g \in R$ a bound $N(g) \in \mathbb{N}$ such that for all $a \in \mathbb{R}^m$ the set $\{y \in \mathbb{R}^n : g(a,y) = 0\}$ has at most $N(g)$ components.

Remark: Following [W1] we define a **component** of a topological space Y to be a closed-and-open subset of Y. The components of Y clearly form a boolean algebra of subsets of Y. Note that an atom of this boolean algebra is necessarily connected. Therefore, if Y has only finitely many components, then the atoms of this boolean algebra are exactly the connected components of Y in the usual sense of "connected component", so Y has then only finitely many connected components.

Proof: Let $g \in R$ and suppose there is no such bound $N(g)$. Then there is for each $j \in \mathbb{N}$ a point $a_j \in \mathbb{R}^m$ such that the subset $\{y \in \mathbb{R}^n \colon g(a_j, y) = 0\}$ of $\mathbb{R}^n$ has pairwise disjoint components $C_{0j}, \ldots, C_{jj}$. Let $\tilde{L}$ be the extension of the language L of ordered rings obtained by including a function symbol f for each function $f \in R$, a unary relation symbol for the subset $\mathbb{N}$ of $\mathbb{R}$, an $(m+1)$-place relation symbol for the relation $\{(j, a_j) \colon j \in \mathbb{N}\}$, and an $(n+2)$-place relation symbol for the relation $\{(i, j, y) \colon y \in C_{ij},\ 0 \le i \le j,\ j \in \mathbb{N}\}$. Let $\tilde{\mathbb{R}}$ be the corresponding expansion of the ordered field of real numbers and let K be a $\mathfrak{c}^+$-saturated elementary extension of $\tilde{\mathbb{R}}$, where $\mathfrak{c}$ is the cardinality of the continuum. Take a nonstandard natural number k in K, that is, $k > n$ for all $n \in \mathbb{N}$ and $\mathsf{K} \models$ "$k \in \mathbb{N}$". Then (the K-interpretation of) each C_{ik} with $i \le k$ and $\mathsf{K} \models$ "$i \in \mathbb{N}$" is a nonempty open-and-closed subset of $Y := \{y \in \mathsf{K}^n \colon g(a_k, y) = 0\}$, in particular C_{ik} is also closed in K^n. Let M be the ring of all functions $y \mapsto f(a_k, y) \colon \mathsf{K}^n \to \mathsf{K}$, for $f \in R$, where of course f is interpreted here as a function on K^{m+n}. Then M is a ring of K-definable C^∞-functions from K^n to K, M contains the coordinate functions $y_1, \ldots, y_n$ on K^n, and M is closed under the operators $\partial/\partial y_j$ for $j = 1, \ldots, n$. Moreover, M is noetherian since it is a homomorphic image of the noetherian ring R. This means we can apply theorem 5.1 from [W1]: for each $i \le k$ with $\mathsf{K} \models$ "$i \in \mathbb{N}$" there are $f_{i1}, \ldots, f_{in} \in M$ such that $C_{ik} \cap V^{\mathrm{reg}}(f_{i1}, \ldots, f_{in}) \ne \emptyset$. But there are at most continuum many possibilities for $f_{i1}, \ldots, f_{in}$, while there are more than continuum many (pairwise disjoint) sets C_{ik}. This gives a contradiction since by the hypothesis of the proposition each set $V^{\mathrm{reg}}(f_{i1}, \ldots, f_{in})$ is finite. ∎

5.10 COROLLARY: *Let $\mathfrak{R}$ be a noetherian H-system and let $f_1, \ldots, f_k$ be a $(P, \mathfrak{R})$-chain on $\mathbb{R}^n$. Let $t_1(x, y), \ldots, t_r(x, y)$ be $L(\mathfrak{R}, f_1, \ldots, f_k)$-terms, where $x = (x_1, \ldots, x_M)$, $y = (y_1, \ldots, y_N)$. Then there is a bound $B = B(t_1, \ldots, t_r) \in \mathbb{N}$ such that for each $a \in \mathbb{R}^M$ the set $V_a(t_1, \ldots, t_r) := \{y \in \mathbb{R}^N \colon t_1(a, y) = \cdots =$*

$t_r(a, y) = 0\}$ has at most B components.

Proof: If some t_i is of the form $t(x, y, \tau(x, y))$ for "simpler" $L(\mathfrak{R}, f_1, \dots, f_k)$-terms $t(x, y, y_{N+1})$ and $\tau(x, y)$, then we can replace the single equation $t_i(x, y) = 0$ by the two equations $t(x, y, y_{N+1}) = \tau(x, y) - y_{N+1} = 0$, and increase N and r by 1. Continuing to unravel terms in this way and increasing N and r (keeping M fixed) we may reduce to the case that for each $i = 1, \dots, r$ the term $t_i(x, y)$ defines either a function on $\mathbb{R}^{M+N}$ that belongs to $\mathfrak{R}_{M+N}$, or $t_i(x, y)$ is of the form $f_j(z_1, \dots, z_n) - z_{n+1}$ where $1 \le j \le k$ and $z_1, \dots, z_n, z_{n+1}$ are distinct variables among $x_1, \dots, x_M, y_1, \dots, y_N$. The function on $\mathbb{R}^{M+N}$ defined by such a term $f_j(z_1, \dots, z_n)$ is easily seen to belong to a $(P, \mathfrak{R})$-chain on $\mathbb{R}^{M+N}$. By concatenating these $(P, \mathfrak{R})$-chains we obtain a single $(P, \mathfrak{R})$-chain $g_1, \dots, g_l$ on $\mathbb{R}^{M+N}$ such that each term $t_i(x, y)$ defines a function on $\mathbb{R}^{M+N}$ that belongs to $\mathfrak{R}_{M+N}\langle g_1, \dots, g_l\rangle$. Since the system $\mathfrak{R}$ is noetherian, the ring $\mathfrak{R}_{M+N}\langle g_1, \dots, g_l\rangle$ is noetherian and closed under the operators $\partial/\partial y_j$. Moreover, the system $\mathfrak{R}\langle g_1, \dots, g_l\rangle^s$ is an H-system by (5.8), so that by corollary (5.5) the hypothesis of (5.9) is satisfied by the ring $\mathfrak{R}_{M+N}\langle g_1, \dots, g_l\rangle$. Then the conclusion of (5.9) applied to the sum of squares of $t_1, \dots, t_r$ gives the desired result. ∎

5.11 Corollary: *Let $\mathfrak{R}$ be a noetherian H-system and $f_1, \dots, f_k$ a $(P, \mathfrak{R})$-chain on $\mathbb{R}^n$. Then we have:*

(i) *Given an existential $L(\mathfrak{R}, f_1, \dots, f_k)$-formula $\phi(x_1, \dots, x_m, x_{m+1})$ there is a bound $C = C(\phi) \in \mathbb{N}$ such that for all $r_1, \dots, r_m \in \mathbb{R}$ the set $\phi(r_1, \dots, r_m, \mathbb{R}) \subseteq \mathbb{R}$ is a union of at most C intervals and C points.*
(ii) *If $T_{\mathfrak{R}, f_1, \dots, f_k}$ is model complete, then $T_{\mathfrak{R}, f_1, \dots, f_k}$ is O-minimal.*

Part (i) extends Cor. 5.4 of [W1] and is an easy consequence of the previous corollary. Part (ii) is immediate from (i).

5.12 Finite Model Completeness. Let $\mathfrak{R}$ be a noetherian system and $f_1, \dots, f_k$ a $(P, \mathfrak{R})$-chain on $\mathbb{R}^n$. Let $\phi(x)$ be an existential $L(\mathfrak{R}, f_1, \dots, f_k)$-formula, $x = (x_1, \dots, x_M)$. Introducing extra existentially quantified variables and arguing as in the proof of (5.10) one obtains an equivalence:

$$T_{\mathfrak{R}, f_1, \dots, f_k} \vdash \phi(x) \leftrightarrow \exists y (f(x, y) = 0),$$

where $y = (y_1, \dots, y_N)$, for some $f \in \mathfrak{R}_{M+N}\langle g_1, \dots, g_l\rangle$ and certain $g_1, \dots, g_l$

forming a $(P, \mathfrak{R})$-chain on $\mathbb{R}^{M+N}$ such that each function g_i is defined by a term $g_i(x,y)$ from $L(\mathfrak{R}, f_1, \dots, f_k)$. With such f and $g_1, \dots, g_l$ we have:

5.13 LEMMA: *There are finitely many N-tuples $h_1 = (h_{11}, \dots, h_{1N}), \dots, h_J = (h_{J1}, \dots, h_{JN})$ with $h_{jr} \in \mathfrak{R}_{M+N}\langle g_1, \dots, g_l\rangle$ such that*

$$T_{\mathfrak{R}, f_1, \dots, f_k} \vdash \exists y (f(x,y) = 0) \leftrightarrow$$
$$\exists y \Big(f(x,y) = 0 \wedge \bigvee_{1 \le j \le J} (h_j(x,y) = 0 \wedge \det(\partial h_j/\partial y)(x,y) \neq 0)\Big),$$

where $(\partial h_j/\partial y)$ is the N-by-N Jacobian matrix $(\partial h_{jr}/\partial y_s)_{1 \le r,s \le N}$.

Proof: Let $\mathsf{K} \models T_{\mathfrak{R}, f_1, \dots, f_k}$ and $a = (a_1, \dots, a_M) \in \mathsf{K}^M$.

Given $h \in \mathfrak{R}_{M+N}\langle g_1, \dots, g_l\rangle$, let $h_a \colon \mathsf{K}^N \to \mathsf{K}$ be given by $h_a(b) = h(a,b)$. Let R_a be the ring of all such functions h_a, so R_a is a noetherian ring of definable C^∞-functions on K^N containing the coordinate functions $y_1, \dots, y_N$ and closed under the operators $\partial/\partial y_s$.

Hence, if $\mathsf{K} \models \exists y (f(a,y) = 0)$, it follows from Theorem 5.1 of [W1] that there are $h_1, \dots, h_N \in \mathfrak{R}_{M+N}\langle g_1, \dots, g_l\rangle$ such that

$$\mathsf{K} \models \exists y (f(a,y) = 0 \wedge h_1(a,y) = \dots = h_N(a,y) = 0 \wedge \det((\partial h_r/\partial y_s)(a,y)) \neq 0).$$

Since this is true for all models K of $T_{\mathfrak{R}, f_1, \dots, f_k}$ and all $a \in \mathsf{K}^N$ the desired conclusion follows by a standard compactness argument. ∎

5.14 COROLLARY: *Let $\mathfrak{R}$ be a noetherian H-system and $f_1, \dots, f_k$ a $(P, \mathfrak{R})$-chain on $\mathbb{R}^n$. If $T_{\mathfrak{R}, f_1, \dots, f_k}$ is model complete, then it is finitely model complete.*

Proof: Given an existential formula $\phi(x)$ with $x = (x_1, \dots, x_M)$, let $f, g_1, \dots, g_l$ be as in (5.12) and take N-tuples $h_1, \dots, h_J \in (\mathfrak{R}_{M+N}\langle g_1, \dots, g_l\rangle)^N$ for which we have an equivalence as in the lemma. By (5.10) there is a bound $B \in \mathbb{N}$, $B \geq 1$, such that for each $\mathsf{K} \models T_{\mathfrak{R}, f_1, \dots, f_k}$ and $a \in \mathsf{K}^M$ we have

$$\operatorname{card}\left(\bigcup_{1 \le j \le J} V^{\mathrm{reg}}(h_{ja})\right) \le B,$$

where h_{ja} denotes the N-tuple $(h_{j1a}, \dots, h_{jNa})$ of C^∞-functions on K^N. Using this fact one easily constructs a quantifier free formula $\theta(x,y)$, such that

$$T_{\mathfrak{R}, f_1, \dots, f_k} \vdash \phi(x) \leftrightarrow \exists y \theta(x,y),$$

and

$$T_{\mathfrak{R}, f_1, \dots, f_k} \vdash \exists^{\le B} y \theta(x,y). \quad ∎$$

6. Proof that $(\mathbb{R}_{\text{an}}, \exp)$ is model complete

6.1 The lemmas in this section are straight forward extensions of results in section 6 and 7 of [W1], and for proofs we can mostly refer to [W1]. We use similar notations to facilitate comparison.

We fix a *noetherian* H-system $\mathfrak{R}$ and a $(P, \mathfrak{R})$-chain $F_1, \dots, F_l$ on $\mathbb{R}^m$.

6.2 Lemma: *Let* $\mathsf{K} \models T_{\mathfrak{R}, F_1, \dots, F_l}$*, let* $r \geq 2$ *and let* $g_1, \dots, g_r\colon \mathsf{K}^r \to \mathsf{K}$ *be given by* $L(\mathfrak{R}, F_1, \dots, F_l, \mathsf{K})$*-terms. Put*

$$V := \{P \in \mathsf{K}^r \colon g_1(P) = \cdots = g_{r-1}(P) = 0\},$$

and suppose that $\det(\partial(g_1, \dots, g_{r-1})/\partial(x_2, \dots, x_r))(P) \neq 0$ *for all* $P \in V$*. Then there is a finite set* $\mathfrak{S}$ *of pairs* (I, ϕ) *such that:*

(i) *The first component* I *of each pair* $(I, \phi) \in \mathfrak{S}$ *is an open interval in* K *and the second component* $\phi\colon I \to \mathsf{K}^{r-1}$ *is a definable* C^∞*-map.*

(ii) *For each* $(I, \phi) \in \mathfrak{S}$*, if* $\sup(I) \in \mathsf{K}$*, then* $\|\phi(x)\| \to \infty$ *as* $x \to \sup(I)$*, and similarly if* $\inf(I) \in \mathsf{K}$*.*

(iii) V *is the disjoint union of the graphs* $\Gamma(\phi)$ *for* $(I, \phi) \in \mathfrak{S}$*.*

Proof: See the proof of Theorem 6.2 in [W1]. ∎

Besides the originally given $(P, \mathfrak{R})$-chain $F_1, \dots, F_l$ it is useful to consider also related $(P, \mathfrak{R})$-chains:

6.3 Definition. An (n, r)-sequence (relative to $(\mathfrak{R}, F_1, \dots, F_l)$) is a sequence $\sigma = (\sigma_1, \dots, \sigma_n)$ of $L(\mathfrak{R}, F_1, \dots, F_l)$-terms whose variables are among $x_1, \dots, x_r$, such that the functions $f_1, \dots, f_n\colon \mathbb{R}^r \to \mathbb{R}$ defined by $\sigma_1, \dots, \sigma_n$ respectively form a $(P, \mathfrak{R})$-chain on $\mathbb{R}^r$. (This notion of (n, r)-sequence is related to Wilkie's.) Clearly an (n, r)-sequence is also an (n, s)-sequence for each $s \geq r$.

6.4 Definition. Given an (n, r)-sequence $\sigma = (\sigma_1, \dots, \sigma_n)$, a model $\mathsf{K} \models T_{\mathfrak{R}, F_1, \dots, F_l}$ and a tuple $a = (a_1, \dots, a_M) \in \mathsf{K}^M$ we define $R^r(a, \mathsf{K}, \sigma)$ to be the ring of all functions $f\colon \mathsf{K}^r \to \mathsf{K}$ for which there is $p \in \mathfrak{R}_{M+r+n}$ such that $f(x) = p(a, x, \sigma_1(x), \dots, \sigma_n(x))$ for all $x = (x_1, \dots, x_r) \in \mathsf{K}^r$.

Note that then $R^r(a, \mathsf{K}, \sigma)$ is a noetherian ring of definable C^∞-functions on K^r closed under the operators $\partial/\partial x_i$ for $i = 1, \dots, r$. If $M \leq N$ and $a \in \mathsf{K}^M$ is a subsequence of $b \in \mathsf{K}^N$, then $R^r(a, \mathsf{K}, \sigma) \subseteq R^r(b, \mathsf{K}, \sigma)$.

Given a substructure $\mathsf{k} \subseteq \mathsf{K}$ we let $R^r(\mathsf{k}, \mathsf{K}, \sigma)$ be the union of the rings $R^r(a, \mathsf{K}, \sigma)$ over all $a \in \mathsf{k}^M$, $M \in \mathbb{N}$. Then $R^r(\mathsf{k}, \mathsf{K}, \sigma)$ is also a (possibly not noetherian) ring of definable C^∞-functions on K^r closed under the operators $\partial/\partial x_i$ for $i = 1, \ldots, r$. It contains the functions on K^r given by the constants in k, the elements of $\mathfrak{R}_r$, and the terms $\sigma_1, \ldots, \sigma_n$.

6.5 Definition. Let σ be an (n, r)-sequence and $\mathsf{k} \subseteq \mathsf{K} \models T_{\mathfrak{R}, F_1, \ldots, F_l}$. Then a point $P \in \mathsf{K}^r$ is called (k, σ)-**definable** if there are $g_1, \ldots, g_r \in R^r(\mathsf{k}, \mathsf{K}, \sigma)$ such that $P \in V^{\text{reg}}(g_1, \ldots, g_r)$.

By routine arguments (see (3.8), (3.9) and the proof of (5.9)) we get:

6.6 Lemma: *$T_{\mathfrak{R}, F_1, \ldots, F_l}$ is model complete if and only if for all models k and K of $T_{\mathfrak{R}, F, \ldots, F}$ with $\mathsf{k} \subseteq \mathsf{K}$ and all (n, r)-sequences σ $(n \geq 0, r \geq 1)$, each (k, σ)-definable point of K^r lies in k^r.*

Remark: In this paper we apply the general results on an arbitrary $(P, \mathfrak{R})$-chain $F_1, \ldots, F_l$ only to the particular $(P, \mathfrak{R})$-chain of length 1 on $\mathbb{R}$ that consists of the single function exp. For that reason we do not actually need lemma (6.6), and will use instead lemma (3.9), which is more convenient in that case. We also point out that it suffices to consider in lemma (6.6) (n, r)-sequences $(\sigma_1, \ldots, \sigma_n)$ such that each σ_i is of the form $F_j(y_1, \ldots, y_m)$ where $y_1, \ldots, y_m$ are distinct variables among $x_1, \ldots, x_r$. ∎

6.7 Lemma: *Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an (n, r)-sequence, $r \geq 2$, and let k and K be models of $T_{\mathfrak{R}, F_1, \ldots, F_l}$ with $\mathsf{k} \subseteq \mathsf{K}$. Assume in addition to the hypotheses of lemma (6.2) that $g_1, \ldots, g_{r-1} \in R^r(\mathsf{k}, \mathsf{K}, \sigma)$ and that each (k, σ)-definable point of $V \subseteq \mathsf{K}^r$ lies in k^r. Let $(\alpha, P) \in V$ be a k-bounded point, $\alpha \in \mathsf{K}$, $P \in \mathsf{K}^{r-1}$. Then there are γ_1, γ_2, β_1, β_2, B_1, B_2 in k with $\gamma_2 < \gamma_1 < \alpha < \beta_1 < \beta_2$ and $\|P\| < B_1 < B_2$, and an integer $M \geq 1$ and definable C^∞-maps ϕ_i: $(\gamma_2, \beta_2) \to \mathsf{K}^{r-1}$ (for $i = 1, \ldots, M$) such that*

(i) *$\|\phi_i(t)\| < B_1$ for $i = 1, \ldots, M$ and $t \in (\gamma_2, \beta_2)$,*

(ii) *$V \cap ((\gamma_2, \beta_2) \times \{Q \in \mathsf{K}^{r-1}\colon \|Q\| < B_2\})$ is the disjoint union of the graphs $\Gamma(\phi_i)$ for $i = 1, \ldots, M$.*

Moreover, for any such γ_1, γ_2, β_1, β_2, B_1, B_2 and M there exist C^∞-maps Ψ_i: $(\gamma_2, \beta_2)_{\mathsf{k}} \to \mathsf{k}^{r-1}$, definable in k, for $i = 1, \ldots, M$, such that (i) and (ii) hold with Ψ_i in place of ϕ_i, where all notions are interpreted in k.

Remark: It also follows from the additional assumption that each point of V with first coordinate in k lies in k^r.

Proof: Just like the proof of lemma 6.3 in [W1], except that theorem 4.9 of [W1] should not be applied to $R^r(\mathsf{k},\mathsf{K},\sigma)$ (since we don't know if this ring is noetherian) but to a suitable (noetherian) subring $R^r(a,\mathsf{K},\sigma)$. The role of theorem 6.2 of [W1] is taken over by lemma (6.2) above. ∎

6.8 LEMMA: *Suppose $T_{\mathfrak{R}}$ is model complete. (Hence O-minimal by (5.11).) Let k and K be models of $T_{\mathfrak{R},F_1,\ldots,F_l}$ with $\mathsf{k}\subseteq\mathsf{K}$. If σ is an (n,r)-sequence, $r\geq 1$, such that for each $s\geq r$ each (k,σ)-definable point of K^s is k-bounded, then each (k,σ)-definable point of K^r lies in k^r.*

Proof: By induction on n. The case $n=0$ follows from model completeness of $T_{\mathfrak{R}}$. The proof is otherwise along the lines of the proof of lemma 2.8 in section 7 of [W1], and is in fact smoother and shorter, since there are no problems with domains of definition as in [W1]. Of course the role of lemma 6.3 in [W1] is taken over by lemma (6.7) above. ∎

6.9 THEOREM: $(\mathbb{R}_{\mathrm{an}},\exp)$ *is finitely model complete and O-minimal.*

Proof: Apply the above to the noetherian H-system

$$\mathfrak{R} := (\hat{\mathbb{R}}\{X,m\}[x_1,\ldots,x_m])_{m\in\mathbb{N}}$$

and the $(P,\mathfrak{R})$-chain of length 1 on $\mathbb{R}$ consisting just of the function exp. Proposition (4.5) (see also (4.4)) shows that the hypothesis of (6.8) is satisfied in this situation for the (r,r)-sequence $(\exp(x_1),\ldots,\exp(x_r))$, for each $r\geq 1$. Hence the desired result follows from (6.8), (3.9), (5.11) and (5.14). ∎

If one analyzes what is really used in this proof one obtains:

6.10 PROPOSITION: *Let $\mathfrak{R}$ be a noetherian H-system such that $\mathfrak{R}_1$ contains the functions $x\mapsto(1+x^2)^{-1}\colon\mathbb{R}\to\mathbb{R}$, and $x\mapsto\exp((1+x^2)^{-1})$. Suppose $T_{\mathfrak{R}}$ is model complete and of rational type. Then $T_{\mathfrak{R},\exp}$ is finitely model complete and O-minimal.*

We now apply this proposition to show that expanding the model complete structures of [W1] by exp preserves model completeness.

6.11 Let a Pfaffian chain $G_1, \ldots, G_l$: $U \to \mathbb{R}$ on an open set $U \subseteq \mathbb{R}^m$ with $m \geq 1$ and $[0,1]^m \subseteq U$ be given, that is, the G's are real analytic functions, and there are polynomials $p_{ij} \in \mathbb{R}[x_1, \ldots, x_{m+i}]$ for $i = 1, \ldots, l, j = 1, \ldots, m$ such that $(\partial G_i/\partial x_j)(x) = p_{ij}(x, G_1(x), \ldots, G_i(x))$ on U.

Let C be a subfield of $\mathbb{R}$ containing all coefficients of all polynomials p_{ij}, for instance the field generated by these coefficients. Define F_i: $\mathbb{R}^m \to \mathbb{R}$ for $i = 1, \ldots, m$ by

$$F_i(x) = \begin{cases} G_i(x), & \text{if } x \in [0,1]^m \\ 0, & \text{if } x \in \mathbb{R}^m[0,1]^m. \end{cases}$$

Then one of the main results of [W1] is (slightly reformulated):

$$(\mathbb{R}, <, (c)_{c \in C}, -, +, \cdot, F_1, \ldots, F_l) \quad \text{is model complete.}$$

To show that further expansion of this structure by exp preserves (finite) model completeness, we now construct a suitable noetherian H-system. Given any subset s of $\{1, \ldots, m\}$, define $F_{i,s}$: $\mathbb{R}^m \to \mathbb{R}$ by $F_{i,s}(x) = F_i(x'_1, \ldots, x'_m)$, where

$$x'_j = \begin{cases} 0, & \text{for } j \in s \\ (1 + x_j^2)^{-1}, & \text{for } j \in \{1, \ldots, m\} \backslash s. \end{cases}$$

The functions $(1+x_1^2)^{-1}, \ldots, (1+x_m^2)^{-1}$ on $\mathbb{R}^m$ followed by a suitable arrangement of the functions $F_{i,s}$, ($i \in \{1, \ldots, l\}$ and $s \subseteq \{1, \ldots, m\}$) are easily seen to form a Pfaffian chain $g_1, \ldots, g_L$ on $\mathbb{R}^m$ whose corresponding polynomials have coefficients in C.

Given any $n \in \mathbb{N}$, sequence $\alpha = (\alpha(1), \ldots, \alpha(m)) \in \{1, \ldots, n\}^m$ and function g_j of this chain we define $g_{j,\alpha}$: $\mathbb{R}^n \to \mathbb{R}$ by $g_{j,\alpha}(x_1, \ldots, x_n) = g_j(x_{\alpha(1)}, \ldots, x_{\alpha(m)})$, and put

$$\mathfrak{R}_n = C[x_1, \ldots, x_n, (g_{j,\alpha})] \quad (j \text{ ranging over } \{1, \ldots, L\} \text{ and } \alpha \text{ over } \{1, \ldots, n\}^m),$$

a noetherian subring of the ring of all C^∞-functions on $\mathbb{R}^n$. One checks easily that $\mathfrak{R} := (\mathfrak{R}_n)$ is a system of C^∞-rings, and that the $T_{\mathfrak{R}}$-expansion of the ordered field of reals is interdefinable, both existentially and universally, with the original structure $(\mathbb{R}, <, (c)_{c \in C}, -, +, \cdot, F_1, \ldots, F_l)$. Hence $T_{\mathfrak{R}}$ is model complete and O-minimal, so $\mathfrak{R}$ is a noetherian H-system. We can now draw the following conclusions.

6.12 COROLLARY:

(i) $(\mathbb{R}, <, (c)_{c\in C}, -, +, \cdot, F_1, \ldots, F_l)$ *is finitely model complete and smooth.*

(ii) $(\mathbb{R}, <, (c)_{c\in C}, -, +, \cdot, F_1, \ldots, F_l, \exp))$ *is finitely model complete and O-minimal.*

Proof: The functions in $\mathfrak{R}$ are "uniquely" existentially definable in

$$(\mathbb{R}, <, (c)_{c\in C}, -, +, \cdot, F_1, \ldots, F_l),$$

hence finite model completeness of this structure follows from finite model completeness of $T_{\mathfrak{R}}$, which in turn follows from (5.14). Smoothness is obtained in the same way as smoothness of T_e, as in [W2], see also the proof of (2.2). Now (ii) is obtained from (i) by applying (6.10): note that we can always extend the chain $F_1, \ldots, F_l$ if necessary to include the functions $x \mapsto (1+x^2)^{-1}$ and $x \mapsto \exp((1+x^2)^{-1})$ in $\mathfrak{R}_1$. ∎

7. Characterization of definable closures

In this short section we keep the notations of the previous section: $\mathfrak{R}$ is a fixed noetherian H-system and $F_1, \ldots, F_l$ a $(P, \mathfrak{R})$-chain on $\mathbb{R}^m$.

Definition: Given an L-structure K and a subset S of K we say that an element $c \in \mathsf{K}$ is **existentially definable over** S if there is an existential $L(S)$-formula $\phi(y)$ in one free variable y such that $\mathsf{K} \models \phi(c)$ and $\mathsf{K} \models \neg\phi(b)$ for all $b \in \mathsf{K}$ with $b \neq c$. The **existential-definable closure of** S **in** K is the set of all elements of K that are existentially definable over S. (Clearly this existential-definable closure is the underlying set of a substructure of K.) ∎

If $\mathrm{Th}(\mathsf{K})$ is model complete, "existentially definable" equals "definable", and the existential-definable closure of S in K equals its definable closure in K, and in this case the definable closure is also the underlying set of the smallest elementary submodel of K containing S. In particular, if K is a real closed field, the definable closure of S in K equals the relative algebraic closure in K of the field generated by S in K. When $T_{\mathfrak{R},F_1,\ldots,F_l}$ is model complete and $\mathsf{K} \models T_{\mathfrak{R},F_1,\ldots,F_l}$ we want to find a similarly simple characterization of the definable closure of S in K. We may as well characterize instead the existential-definable closure of S in K, which has the advantage that we do not have to assume $T_{\mathfrak{R},F_1,\ldots,F_l}$ is model complete. The answer is not as simple as for real closed fields, but still perhaps of interest.

PROPOSITION 7.1: *Let* $\mathsf{K} \models T_{\mathfrak{R},F_1,\ldots,F_l}$ *and* $S \subseteq \mathsf{K}$. *Let* $c \in \mathsf{K}$. *Then* c *is existentially definable over* S *if and only if* c *is a coordinate of a regular solution (in* K^M *for some* $M \in \mathbb{N}$*) of a system of equations*

$$t_1(x_1, \ldots, x_M) = \cdots = t_M(x_1, \ldots, x_M) = 0$$

with $t_1, \ldots, t_M$ *terms of* $L(\mathfrak{R}, F_1, \ldots, F_l, S)$. *Moreover, the existential-definable closure of* S *in* K *is existentially closed in* K.

Proof: Such a system of equations has only finitely many regular solutions, and by lexicographically ordering these solutions we can tell them apart, and this enables us to existentially define each coordinate of each regular solution. For the reverse inclusion, let $E(S)$ be the set of $a \in \mathsf{K}$ that are coordinates of regular solutions of systems as above. We claim that if a, $b \in E(S)$, then also $a + b \in E(S)$. To see this, let a be the i-th coordinate of a regular solution of the system $t_1(x) = \cdots = t_M(x) = 0$ with $x = (x_1, \ldots, x_M)$, and b the j-th coordinate of a regular solution of the system $u_1(y) = \cdots = u_N(y) = 0$ with $y = (y_1, \ldots, y_N)$, and with the t's and u's terms of $L(\mathfrak{R}, F_1, \ldots, F_l, S)$. We may as well assume the variables $x_1, \ldots, x_M, y_1, \ldots, y_N$ are distinct, and take one extra variable z. Then $a+b$ is clearly the $(M+N+1)$-th coordinate of a regular solution in K^{M+N+1} of the system

$$t_1(x) = \cdots = t_M(x) = u_1(y) = \cdots = u_N(y) = z - (x_i + y_j) = 0.$$

In the same way one shows $ab \in E(S)$, and even that $E(S)$ is (the underlying set of) a substructure of the $L(\mathfrak{R}, F_1, \ldots, F_l)$-structure K. The proof will now be finished by showing that $E(S)$ is existentially closed in K, since this clearly implies that $E(S)$ is the existential-definable closure of S in K. By the considerations of (5.12) and lemma (5.13) it suffices to show:

CLAIM: *Let* $h_1(y), \ldots, h_N(y)$ *be* $L(\mathfrak{R}, F_1, \ldots, F_l, E(S))$*-terms,* $y = (y_1, \ldots, y_N)$, *and let* $(b_1, \ldots, b_N) \in \mathsf{K}^N$ *be a regular solution to the system* $h_1(y) = \cdots = h_N(y) = 0$. *Then* $b_j \in E(S)$ *for all* j .

To prove this claim, write each $h_j(y)$ as $t_j(a, y)$ where $t_1(x, y), \ldots, t_N(x, y)$ are $L(\mathfrak{R}, F_1, \ldots, F_l)$-terms, $x = (x_1, \ldots, x_M)$ and $a = (a_1, \ldots, a_M) \in E(S)^M$. For each $i \in \{1, \ldots, M\}$, let $z_i = (z_{i1}, \ldots, z_{in(i)})$ be a tuple of new variables and let $t_{i1}(z_i), \ldots, t_{in(i)}(z_i)$ be $L(\mathfrak{R}, F_1, \ldots, F_l, S)$-terms such that a_i is a coordinate of a regular solution in $\mathsf{K}^{n(i)}$ of the system $t_{i1}(z_i) = \cdots = t_{in(i)}(z_i) = 0$, say the

$m(i)$-th coordinate of such a regular solution, $1 \leq m(i) \leq n(i)$. Let $n = \sum n(i)$. We now consider the following system of $M + N + n$ equations in the variables $x_1, \ldots, x_M, y_1, \ldots, y_N, z_{11}, \ldots, z_{1n(1)}, \ldots, z_{M1}, \ldots, z_{Mn(M)}$:

$$\begin{aligned}
&t_1(x,y) = \cdots = t_N(x,y) = 0,\\
&t_{11}(z_1) = \cdots = t_{1n(1)}(z_1) = 0\\
&t_{21}(z_2) = \cdots = t_{2n(2)}(z_2) = 0\\
&\vdots\\
&t_{M1}(z_M) = \cdots = t_{Mn(M)}(z_M) = 0\\
&x_1 - z_{1m(1)} = \cdots = x_M - z_{Mm(M)} = 0.
\end{aligned}$$

Note that the number of variables of this system is also $M + N + n$. This system has a regular solution of the form $(a_1, \ldots, a_M, b_1, \ldots, b_N, \ldots)$. Hence each $b_i \in E(S)$. ∎

8. Analytic Cell Decomposition

8.1 We call a system $\mathfrak{R}$ **analytic** if all functions in $\mathfrak{R}$ are (real) analytic.

In this section we show that if $F_1, \ldots, F_l$ is a $(P, \mathfrak{R})$-chain over an analytic system $\mathfrak{R}$, then $F_1, \ldots, F_l$ are themselves analytic, and the usual cell decomposition theorem can be improved to give analytic cells, when we assume also that $\mathfrak{R}$ is noetherian and $T_{\mathfrak{R}, F_1, \ldots, F_l}$ is model complete. First a purely analytic result.

8.2 LEMMA: *Let $f\colon U \to \mathbb{R}$ be a C^1-function on an open set $U \subseteq \mathbb{R}^m$ such that $\partial f / \partial x_i = p_i(x, f(x))$ on U for $i = 1, \ldots, m$, where the $p_i\colon V \to \mathbb{R}$ are analytic functions on an open set $V \subseteq \mathbb{R}^{m+1}$ that contains the graph of f. Then f is analytic.*

Proof: Let $m \geq 1$ and assume inductively that the lemma holds for m replaced by $m - 1$. Since analyticity is a local property, we may take $U = I_1 \times \cdots \times I_m$, with open intervals $I_1, \ldots, I_m$, and that $V = U \times J$, where J is also an open interval. Take a point $r \in I_m$ and define the C^1-function $g\colon I_1 \times \cdots \times I_{m-1} \to \mathbb{R}$ by $g(x_1, \ldots, x_{m-1}) = f(x_1, \ldots, x_{m-1}, r)$. Then the inductive hypothesis implies that g is analytic. Next we use the **analytic dependence on parameters and initial values of solutions of analytic ordinary differential equations**, which for the reader's convenience we state in a global form as follows :

FACT: *Let $p\colon W \times I \times J \to \mathbb{R}$ be analytic, with W open in $\mathbb{R}^n$ and I, J open intervals, and fix $r \in I$. For each $a \in W$ and $b \in J$, consider the collection $C_{a,b}$ of C^1-functions $h\colon I' \to \mathbb{R}$ such that I' is an open subinterval of I containing r, the graph of h is contained in $I \times J$, $h'(x) = p(a, x, h(x))$ for all $x \in I'$, and $h(r) = b$. Then $C_{a,b}$ contains a "maximal solution" $y_{a,b}\colon I_{a,b} \to \mathbb{R}$, that is, $I_{a,b}$ contains every interval I' as above and $y_{a,b}$ extends every function $h \in C_{a,b}$. Moreover, $D := \{(a, b, x)\colon a \in W, b \in J, x \in I_{a,b}\}$ is an open subset of $\mathbb{R}^{n+2}$ and $y\colon (a, b, x) \to y_{a,b}(x)\colon D \mapsto \mathbb{R}$ is analytic.* .

Now apply this to $W := I_1 \times \cdots \times I_{m-1}$, $I := I_m$, $J := J$ and $p := p_m$. Then clearly $f(x_1, \ldots, x_{m-1}, x_m) = y(x_1, \ldots, x_{m-1}, g(x_1, \ldots, x_{m-1}), x_m)$ in the notation above. Hence the analyticity of f follows from the analyticity of g and y. ∎

8.3 COROLLARY: *If $\mathfrak{R}$ is an analytic system, then the functions in any $(P, \mathfrak{R})$-chain over any $\mathbb{R}^m$ are analytic.*

Proof: By induction on the length of the chain, using the lemma above. ∎

Note that the systems of (3.2) and those introduced in (6.11) are all analytic. Also, if $\mathfrak{R}$ is analytic and $f_1, \ldots, f_k$ is a $(P, \mathfrak{R})$-chain over $\mathbb{R}^m$, then the corollary implies that $\mathfrak{R}\langle f_1, \ldots, f_k\rangle^s$ is analytic.

8.4 Let $\tilde{\mathbb{R}}$ be an expansion of the ordered field of reals. A map $f\colon A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^m$ is called **$\tilde{\mathbb{R}}$-analytic** if A is definable and there is a definable open neighborhood U of A in $\mathbb{R}^m$ and a definable real analytic map $F\colon U \to \mathbb{R}^n$ such that $f = F|A$. *Here and in the rest of this section we take* "definable" *in its absolute sense, that is* "definable in $\tilde{\mathbb{R}}$ without constants". In particular, an $\tilde{\mathbb{R}}$-analytic map is definable.

Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be definable and $f = (f_1, \ldots, f_n) : A \to \mathbb{R}^n$. Then:

(i) the inclusion map $A \to \mathbb{R}^m$ is $\tilde{\mathbb{R}}$-analytic;

(ii) each coordinate map $(x_1, \ldots, x_m) \to x_i\colon A \to \mathbb{R}$ is $\tilde{\mathbb{R}}$-analytic;

(iii) f is $\tilde{\mathbb{R}}$-analytic if and only if each f_i is $\tilde{\mathbb{R}}$-analytic;

(iv) if f is $\tilde{\mathbb{R}}$-analytic and $g\colon B \to \mathbb{R}^k$ is $\tilde{\mathbb{R}}$-analytic, then the composition $g \circ f\colon A \cap f^{-1}(B) \to \mathbb{R}^k$ is $\tilde{\mathbb{R}}$-analytic.

8.5 We define $\tilde{\mathbb{R}}$-analytic cells in $\mathbb{R}^n$ as certain kinds of definable subsets of $\mathbb{R}^n$; the definition is by induction on n:

(i) the $\tilde{\mathbb{R}}$-analytic cells in $\mathbb{R} = \mathbb{R}^1$ are just the definable points $\{r\}$ and the definable open intervals (a, b), $-\infty \leq a < b \leq +\infty$;

(ii) let $C \subseteq \mathbb{R}^n$ be an $\tilde{\mathbb{R}}$-analytic cell and let f, g: $C \to \mathbb{R}$ be $\tilde{\mathbb{R}}$-analytic functions such that $f < g$ on C; then $(f,g) := \{(x,r) \in C \times \mathbb{R}\colon f(x) < r < g(x)\}$ is an $\tilde{\mathbb{R}}$-analytic cell in $\mathbb{R}^{n+1}$; also, the graph $\Gamma(f) \subseteq C \times \mathbb{R}$ and the sets $(-\infty, f) := \{(x,r) \in C \times \mathbb{R}\colon r < f(x)\}$, $(g, +\infty) := \{(x,r) \in C \times \mathbb{R}\colon g(x) < r\}$ and $(-\infty, +\infty) := C \times \mathbb{R}$ are $\tilde{\mathbb{R}}$-analytic cells in $\mathbb{R}^{n+1}$.

In this way all $\tilde{\mathbb{R}}$-analytic cells are obtained. Note that an $\tilde{\mathbb{R}}$-analytic cell in $\mathbb{R}^n$ is a real analytic submanifold of $\mathbb{R}^n$, definably-analytically isomorphic to $\mathbb{R}^m$ for some $m \leq n$.

8.6 An $\tilde{\mathbb{R}}$-analytic decomposition of $\mathbb{R}^n$ is a special kind of partition of $\mathbb{R}^n$ into finitely many $\tilde{\mathbb{R}}$-analytic cells. Definition is by induction on n:

(i) An $\tilde{\mathbb{R}}$-analytic decomposition of $\mathbb{R}^1 = \mathbb{R}$ is a collection of intervals and points of the form $\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}$ with $a_1 < \cdots < a_k$ definable real numbers. (For $k = 0$ this is $\{\mathbb{R}\}$.)

(ii) An $\tilde{\mathbb{R}}$-analytic decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into $\tilde{\mathbb{R}}$-analytic cells A such that the set of projections $\pi(A)$ is an $\tilde{\mathbb{R}}$-analytic decomposition of $\mathbb{R}^n$. (Here π: $\mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.)

An $\tilde{\mathbb{R}}$-analytic decomposition of $\mathbb{R}^n$ is said to **partition** a set $A \subseteq \mathbb{R}^n$ if A is a union of cells in the decomposition.

8.7 Let now $\mathfrak{R}$ be a noetherian H-system and $F_1, \ldots, F_l$ a $(P, \mathfrak{R})$-chain on $\mathbb{R}^m$. Let $\tilde{\mathbb{R}}$ be the corresponding $L(\mathfrak{R}, F_1, \ldots, F_l)$-expansion of the ordered field of reals, so $\tilde{\mathbb{R}}$ is a model of $T_{\mathfrak{R}, F_1, \ldots, F_l}$. Then the usual Cell Decomposition Theorem for O-minimal structures, cf. [P-S], can be refined as follows:

8.8 THEOREM: *Assume $\mathfrak{R}$ is analytic and $T_{\mathfrak{R}, F_1, \ldots, F_l}$ is model complete. Then:*

(I_n) *For any definable sets $A_1, \ldots, A_k \subseteq \mathbb{R}^n$ there is an $\tilde{\mathbb{R}}$-analytic decomposition of $\mathbb{R}^n$ partitioning $A_1, \ldots, A_k$.*

(II_n) *For every definable function f: $A \to \mathbb{R}, A \subseteq \mathbb{R}^n$, there is an $\tilde{\mathbb{R}}$-analytic decomposition of $\mathbb{R}^n$ partitioning A such that each restriction $f|C$: $C \to \mathbb{R}$ is $\tilde{\mathbb{R}}$-analytic for each cell $C \subseteq A$ in the decomposition.*

Proof: By induction on n. Note first that by (5.11)(ii) the theory we are dealing with is O-minimal. Ordinary cell decomposition then gives (I_1). For (II_1), let f: $A \to \mathbb{R}$ be definable with $A \subseteq \mathbb{R}$. Model completeness gives an equivalence: $\tilde{\mathbb{R}} \models (x,y) \in \Gamma(f) \leftrightarrow \exists z (F(x,y,z) = 0)$, $(z = (z_1, \ldots, z_N))$ for some function

$F \in \mathfrak{R}_{2+N}\langle g_1, \ldots, g_l \rangle$ where $g_1, \ldots, g_l$ is a $(P, \mathfrak{R})$-chain on $\mathbb{R}^{2+N}$ such that each g_i is defined by an $L(\mathfrak{R}, F_1, \ldots, F_l)$-term. By lemma (5.13) and corollary (8.3) there are finitely many $(1+N)$-tuples

$$h_1 = (h_{11}, \ldots, h_{11+N}), \ldots, h_J = (h_{J1}, \ldots, h_{J1+N})$$

of $\tilde{\mathbb{R}}$-analytic functions h_{jr} on $\mathbb{R}^{2+N}$ such that:

$$\begin{aligned} &\tilde{\mathbb{R}} \models \exists y, z(F(x,y,z) = 0) \\ \leftrightarrow &\bigvee_{1 \le j \le J} \exists y, z(F(x,y,z) = 0 \wedge h_j(x,y,z) = 0 \wedge \det(\partial h_j/\partial(y,z))(x,y,z) \ne 0). \end{aligned}$$

Thus $A = \bigcup_{1 \le j \le J} A_j$, where A_j is the set of $a \in A$ such that

$$\tilde{\mathbb{R}} \models \exists z(h_j(a, f(a), z) = 0 \wedge \det(\partial h_j/\partial(y,z))(a, f(a), z) \ne 0).$$

Applying ordinary cell decomposition to the restrictions of f to the A_j's we may reduce to the case that $J = 1$, A is an interval (i.e., an open $\tilde{\mathbb{R}}$-analytic cell in $\mathbb{R}^1$) and f is continuous. Then the analytic implicit function theorem implies f is analytic, so f is $\tilde{\mathbb{R}}$-analytic. Next assume inductively that $(I_1), \ldots, (I_n)$, $(II_1), \ldots, (II_n)$ hold. Then the usual O-minimal cell decomposition theorem, together with the inductive hypothesis easily gives (I_{n+1}). (One may consult the proof of the cell decomposition theorem in [P-S] for more details.) Next one derives (II_{n+1}) in almost the same fashion as we derived (II_1) from (I_1). Again, we refer to [P-S] for similar arguments of this kind. ∎

8.9 Remark. What if we consider sets and functions that are definable using constants? In practice this is no problem since usually $\mathfrak{R}_0 = \mathbb{R}$ so that there is then no difference between "definable in the absolute sense" and "definable using constants from $\mathbb{R}$". Even if $\mathfrak{R}_0$ does not contain all constants, we can usually extend $\mathfrak{R}$ so that $\mathfrak{R}_0 = \mathbb{R}$.

8.10 Let us consider the 1-variable case in more detail. *Under the assumptions of the theorem, let $g : \mathbb{R} \to \mathbb{R}$ be definable using constants in $\tilde{\mathbb{R}}$.* We claim: There are reals $a_1 < \cdots < a_k$ such that g is analytic on each interval (a_i, a_{i+1}) for $i = 0, \ldots, k$, where $a_0 := -\infty$ and $a_{k+1} := +\infty$.

To see why, note that there are $r_1, \ldots, r_n \in \mathbb{R}$ and a definable function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ such that $g(x) = f(r_1, \ldots, r_n, x)$ for all x, where "definable" is taken here in the absolute sense. Then the claim follows easily from the piecewise analyticity of f implied by the theorem.

9. An exponential bound on the growth of definable functions

9.1 Define $\exp_n(x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$ by $\exp_0(x) := x$ and $\exp_{n+1}(x) := \exp(\exp_n(x))$.

9.2 Proposition: *Let $g: \mathbb{R} \to \mathbb{R}$ be definable (using constants) in $\mathbb{R}_{\exp}$. Then there are $m, n \in \mathbb{N}$ and a constant $C > 0$ such that*

$$|g(x)| < C \cdot \exp_n(x^m)$$

for all sufficiently large x.

Remark: The proof given below actually produces m and n from an algebraic differential equation satisfied by g on some interval $(a, +\infty)$. But first we treat a few generalities on such differentially algebraic functions. ∎

9.3 Let U be a nonempty connected open set in $\mathbb{R}^m$, and let $\mathrm{An}(U)$ be the integral domain of (real) analytic functions $f: U \to \mathbb{R}$. We say that $f \in \mathrm{An}(U)$ is **differentially algebraic** if the integral domain

$$\mathbb{R}[\partial^{|\alpha|} f / \partial x^\alpha \colon \alpha \in \mathbb{N}^m] \subseteq \mathrm{An}(U)$$

generated by the partials of f over the field $\mathbb{R}$ of constant functions on U has finite transcendence degree over $\mathbb{R}$.

(If U is an interval on the real line, this is clearly equivalent to the standard definition that there is a non-zero polynomial $p(X_0, \ldots, X_k)$ over $\mathbb{R}$ such that $p(f(x), f'(x), \ldots, f^{(k)}(x)) = 0$ for all $x \in U$.)

Let $V \subseteq \mathbb{R}^n$ be a nonempty connected open set, $f_1, \ldots, f_n \in \mathrm{An}(U \times V)$. Let $x = (x_1, \ldots, x_m)$ range over U and $y = (y_1, \ldots, y_n)$ over V. Suppose $\eta_1, \ldots, \eta_n: U \to \mathbb{R}$ are continuous such that for all x in U,

$$\eta(x) := (\eta_1(x), \ldots, \eta_n(x)) \in V \ , \ \ f_1(x, \eta(x)) = \cdots = f_n(x, \eta(x)) = 0$$

and $J(x) \neq 0$, where $J(x) := \det((\partial f_i / \partial y_j)(x, \eta(x)))$.

By the analytic implicit function theorem the functions $\eta_1, \ldots, \eta_n$ are then also analytic, and for each $i = 1, \ldots, m$ and $x \in U$:

$$(*) \qquad \begin{pmatrix} \partial\eta_1/\partial x_i \\ \vdots \\ \partial\eta_n/\partial x_i \end{pmatrix} = - \begin{pmatrix} \partial f_1/\partial y_1 \cdots \partial f_1/\partial y_n \\ \vdots \\ \partial f_n/\partial y_1 \cdots \partial f_n/\partial y_n \end{pmatrix}^{-1} \begin{pmatrix} \partial f_1/\partial x_i \\ \vdots \\ \partial f_n/\partial x_i \end{pmatrix}$$

where the left side is evaluated at x and the right side at $(x, \eta(x))$.

9.4 LEMMA: *Suppose $f_1, \dots, f_n$ are differentially algebraic. Then $\eta_1, \dots, \eta_n$ are also differentially algebraic.*

Proof: Let $\mathbb{R}[f_1, \dots, f_n]^d$ be the subring of $\mathrm{An}(U \times V)$ generated over $\mathbb{R}$ by the partials of all orders of $f_1, \dots, f_n$. Note that the function $J\colon U \to \mathbb{R}$ introduced above is real analytic and has no zero on U. Clearly $(*)$ above can be written as

$$(**) \qquad (\partial \eta_j / \partial x_i)(x) = J(x)^{-1} \cdot F_{ji}(x, \eta(x)), \text{ for some } F_{ji} \in \mathbb{R}[f_1, \dots, f_n]^d.$$

By taking further derivatives in $(**)$, using the chain rule, one derives inductively

$$(***) \qquad \text{for each } \alpha \in \mathbb{N}^m : (\partial^{|\alpha|} \eta_j / \partial x^\alpha)(x) = J(x)^{-n(\alpha)} \cdot F_{j\alpha}(x, h(x))$$

for some $n(\alpha) \in \mathbb{N}$ and $F_{j\alpha} \in \mathbb{R}[f_1, \dots, f_n]^d$. Consider now the $\mathbb{R}$-algebra homomorphism $\mathrm{An}(U \times V) \to \mathrm{An}(U)$ sending each function $f \in \mathrm{An}(U \times V)$ to the function $x \mapsto f(x, \eta(x))\colon U \to \mathbb{R}$. Let $R \subseteq \mathrm{An}(U)$ be the image of $\mathbb{R}[f_1, \dots, f_n]^d$ under this homomorphism. Note that $J \in R$. By $(* * *)$ we know that

$$\mathbb{R}[\eta_1, \dots, \eta_n]^d \subseteq R[J^{-1}] \subseteq \mathrm{Frac}(R),$$

where J^{-1} is the multiplicative inverse of J inside the fraction field $\mathrm{Frac}(R)$ of R, which we consider here as a subfield of the fraction field of $\mathrm{An}(U)$. Hence:

$$\mathrm{tr.deg}_{\mathbb{R}} \mathbb{R}[\eta_1, \dots, \eta_n]^d \le \mathrm{tr.deg}_{\mathbb{R}} R[J^{-1}] = \mathrm{tr.deg}_{\mathbb{R}} R \le \mathrm{tr.deg}_{\mathbb{R}} \mathbb{R}[f_1, \dots, f_n]^d < \infty.$$

Therefore $\eta_1, \dots, \eta_n$ are differentially algebraic. ∎

Similar but easier arguments show (with $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ as above):

9.5 LEMMA: *Let $f \in \mathrm{An}(U)$ and $g_1, \dots, g_m \in \mathrm{An}(V)$ be differentially algebraic, such that the image of $g := (g_1, \dots, g_m)\colon V \to \mathbb{R}^m$ is contained in U. Then $f(g_1, \dots, g_m) := f \circ g \in \mathrm{An}(V)$ is differentially algebraic.*

9.6 LEMMA: *If $n = m + 1$ and we assume in lemma (8.2) in addition that the $p_i \in \mathrm{An}(V)$ are differentially algebraic, then f is differentially algebraic.*

9.7 COROLLARY: *Let all functions of the analytic system $\mathfrak{N}$ be differentially algebraic, and let $F_1, \dots, F_l$ be a $(P, \mathfrak{N})$-chain on $\mathbb{R}^m$. Then all functions of $R\langle F_1, \dots, F_l\rangle^s$ are differentially algebraic.*

9.8 PROOF OF PROPOSITION (9.2). We first note that by (8.10) each definable $g\colon \mathbb{R} \to \mathbb{R}$ is analytic on an interval $(a, +\infty)$. The germs at $+\infty$ of such definable functions form of course a subring of the ring of germs at $+\infty$ of all real valued functions on $\mathbb{R}$; this subring is actually a field, since by O-minimality, if $g\colon \mathbb{R} \to \mathbb{R}$ is definable, then $g(x)$ is ultimately of constant sign (positive, negative, or zero) for large x. Moreover, this field is a differential field, since for large x the derivative $g'(x)$ exists, is a definable function of x, and taking the germ at $+\infty$ of this derived function is a well defined operation on germs. This means that the germs at $+\infty$ of definable functions $g\colon \mathbb{R} \to \mathbb{R}$ form, what is called, a Hardy field. Let $\mathfrak{R}$ be the system $(\mathbb{R}[x_1, \ldots, x_n])_{n\in\mathbb{N}}$, so by (9.7) the functions in the extended system $\mathfrak{R}\langle\exp\rangle^s$ are all (analytic and) differentially algebraic. Now we note that the proof of analytic cell decomposition shows that our function g is piecewise "implicitly defined" by equations in $\mathfrak{R}\langle\exp\rangle^s$, so that by (9.7) and (9.4) there is $a \in \mathbb{R}$ such that g is analytic on $(a, +\infty)$, and differentially algebraic on that interval, that is, there is a nonzero polynomial $p(X_0, \ldots, X_n)$ over $\mathbb{R}$ such that $p(g(x), g'(x), \ldots, g^{(n)}(x)) = 0$ for $x > a$. By a general result on differentially algebraic functions in Hardy fields due to M. Singer, cf. [Ro, Th.3], it follows there is an $m \in \mathbb{N}$ and a positive constant C such that $|g(x)| < C \cdot \exp_n(x^m)$ for all sufficiently large x. ∎

9.9 Theorem 3 in [Ro] actually supports a more general result. To see this we need a relative version of "differentially algebraic function". Let $\mathfrak{R}$ be an analytic system; let U be as in (9.3), and call $f \in \mathrm{An}(U)$ **differentially algebraic over** $\mathfrak{R}$ if the subring of $\mathrm{An}(U)$ generated by f and its partials of all orders over $\mathfrak{R}_m|U := \{g|U\colon g \in \mathfrak{R}_m\}$ has finite transcendence degree over $\mathfrak{R}_m|U$. Then the corresponding relative versions of (9.4), (9.5), (9.6) and (9.7) go through. Applying this to $\mathfrak{R} := (\hat{\mathbb{R}}\{X, m\}[x_1, \ldots, x_m])_{m\in\mathbb{N}}$, the proof in (9.8) leads to the following extension of (9.2).

9.10 PROPOSITION: *Let $g\colon \mathbb{R} \to \mathbb{R}$ be definable (using constants) in $(\mathbb{R}_{\mathrm{an}}, \exp)$. Then there are $m, n \in \mathbb{N}$ and a constant $C > 0$ such that $|g(x)| < C \cdot \exp_n(x^m)$ for all sufficiently large x.*

We leave the details of the proof to the reader.

References

[D-vdD] J. Denef and L. van den Dries, *P-adic and real subanalytic sets,* Ann. Math. **128** (1988), 79–138.

[vdD1] L. van den Dries, *Algebraic theories with definable Skolem functions,* Journal of Symbolic Logic **49** (1984), 625–629.

[vdD2] L. van den Dries, *A generalization of the Tarski-Seidenberg theorem, and some nondefinability results,* Bull. AMS **15** (1986), 189–193.

[vdD3] L. van den Dries, *The elementary theory of restricted elementary functions,* J. Symb. Logic **53** (1988), 796–808.

[vdD-M-M] L. van den Dries, A. Macintyre and D. Marker, *The elementary theory of restricted analytic fields with exponentiation,* Annals of Mathematics, to appear.

[F] J. Frisch, *Points de platitude d'un morphisme d'espaces analytiques,* Inv. Math. **4** (1967), 118–138.

[H] A. G. Hovanskii, *On a class of systems of transcendental equations,* Soviet Math. Dokl. **22** (1980), 762–765.

[K-P-S] J. Knight, A. Pillay and C. Steinhorn, *Definable sets in ordered structures. II,* Trans. AMS **295** (1986), 593–605.

[P-S] A. Pillay and C. Steinhorn, *Definable sets in ordered structures. I,* Trans. AMS **295** (1986), 565–592.

[Re] J.-P. Ressayre, *Integer parts of real closed exponential fields,* preprint.

[Ro] M. Rosenlicht, *The rank of a Hardy field,* Trans. AMS **280** (1983), 659–671.

[W1] A. J. Wilkie, *Model completeness results for expansions of the real ordered field I: Restricted Pfaffian functions,* preprint 1991.

[W2] A. J. Wilkie, *Model completeness results for expansions of the real field II: the exponential function,* preprint 1992.